THE GUIDED WAVES MODELLING
-
A SPECTRAL METHOD APPROACH

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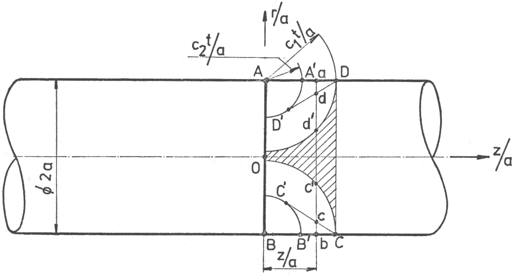
Motivation



Valeš, F., Podélný ráz polonekonečných válcových elastických tyčí kruhového průřezu, část I (Z681/79) a II (Z839/83), ÚT AVČR Praha, (in Czech).

$$\int_0^\infty \frac{(2-\xi^2)\mathrm{J}_1(r\,\gamma a\,B)\mathrm{J}_1(\gamma a\,A) - 2\mathrm{J}_1(\gamma a\,B)\mathrm{J}_1(r\,\gamma a\,A)}{\gamma a\,\xi\,B\,M}\cos(z\,\gamma a)\sin(t\,\xi\,\gamma a)\,d\gamma a,$$

where

$$\begin{aligned} M = & -\gamma a^2\left((2-\xi^2)^2/A + 4\,\kappa\,A\right)\mathrm{J}_0(\gamma a\,B)\,\mathrm{J}_0(\gamma a\,A) \\ & +\gamma a\left((2-\xi^2)\left(4+(2-\xi^2)/(\xi^2-1)\right)+2\,\kappa\,\xi^2\right)\mathrm{J}_0(\gamma a\,B)\,\mathrm{J}_1(\gamma a\,A) \\ & -2\,\gamma a\,(2-\xi^2)\,B/A\,\mathrm{J}_1(\gamma a\,B)\,\mathrm{J}_0(\gamma a\,A) \\ & +\left(2\,B\left(2\,\gamma a^2-(2-\xi^2)/(\xi^2-1)\right)+\kappa\,\gamma a^2\,(2-\xi^2)^2/B\right)\mathrm{J}_1(\gamma a\,B)\,\mathrm{J}_1(\gamma a\,A), \end{aligned}$$

- a radius of the semi-infinite bar,
- γ wavenumber,
- ξ the ratio of the phase velocity and the shear wave velocity, $\xi(\gamma a)$,
- κ the ratio of the squares of the phase velocities,
- A $\sqrt{\kappa\xi^2-1}$,
- B $\sqrt{\xi^2-1}$,
- J the Bessel function of the first kind.

Characteristic function

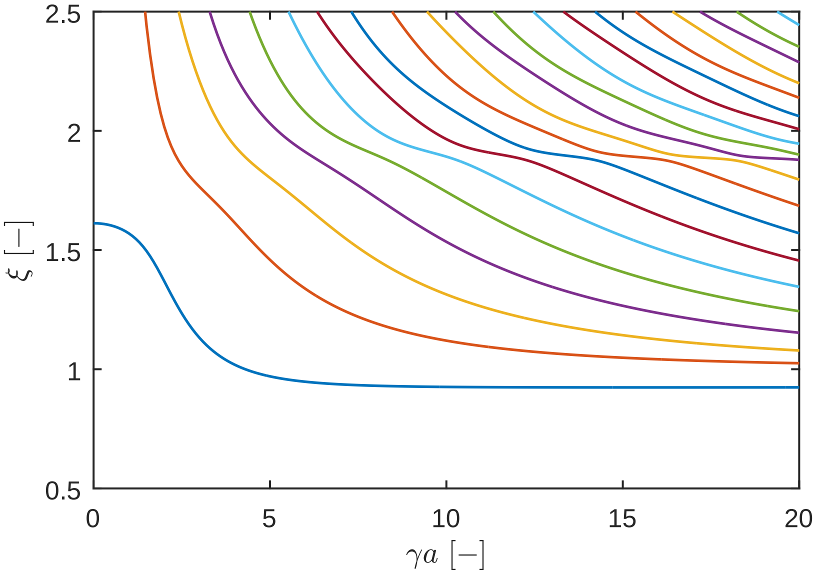
The dispersion relations $f(\xi, \gamma a)$ is defined as

$$\left(2-\xi^2\right)^2 \mathrm{J}_0(\gamma a A)\, \mathrm{J}_1(\gamma a B) + 4AB\mathrm{J}_1(\gamma a A)\, \mathrm{J}_0(\gamma a B) - \frac{2\xi^2}{\gamma a} A\mathrm{J}_1(\gamma a A)\, \mathrm{J}_1(\gamma a B) = 0,$$

where

- a radius of the semi-infinite bar,
- γ wavenumber,
- ξ the ratio of the phase velocity and the shear wave velocity,
- κ the ratio of the squares of the phase velocities,
- A $\sqrt{\kappa\xi^2 - 1}$,
- B $\sqrt{\xi^2 - 1}$,
- J the Bessel function of the first kind.

Dispersion curves



Solution methods

- Root-finding
- Interval arithmetics
  - Pelikán, Hora:
    Robust method for finding of dispersion curves in a thick plate problem,
    COMPUTATIONAL MECHANICS 2014.
- Chebyshev interpolation
  - Hora:
    The root-finding of dispersion curves in a bar impact problem,
    COMPUTATIONAL MECHANICS 2015.
  - Hora:
    The use of the Chebyshev interpolation in elastodynamics problems,
    COMPUTATIONAL MECHANICS 2016.
- Spectral method

Equations of motion

Displacement potentials:

$$
\begin{aligned}
u_r &= \partial_r \phi - \partial_z \psi_\theta, \\
u_z &= \partial_z \phi + r^{-1} \partial_r (r \psi_\theta).
\end{aligned}
$$

The equations of axisymmetric motion:

$$
\begin{aligned}
\nabla^2 \phi &= \frac{1}{c_1^2} \partial_t^2 \phi, \\
\left(\nabla^2 - \frac{1}{r^2} \right) \psi_\theta &= \frac{1}{c_2^2} \partial_t^2 \psi_\theta,
\end{aligned}
$$

where

- c_1 the P-wave velocity,
- c_2 the S-wave velocity,
- ∇^2 Laplace operator, $\nabla^2 = \partial_r^2 + r^{-1} \partial_r + \partial_z^2$.

The normal and tangential stress tractions:

$$\sigma_{rr} = \lambda\Delta + 2\mu\partial_r u_r,$$
$$\sigma_{rz} = \mu(\partial_z u_r + \partial_r u_z),$$

where

Δ the dilatation in cylindrical $r - z$ coordinates,

λ, μ the Lame parameters.

The propagation of sinusoidal waves:

$$\phi = \Phi e^{i(k_z z - \omega t)}, \qquad \psi_\theta = \Psi e^{i(k_z z - \omega t)},$$

where

ω the angular frequency,

k_z the axial wave number,

Φ, Ψ the amplitudes.

Helmholtz equations:

$$\underbrace{\left(\mathrm{d}_r^2 + r^{-1}\mathrm{d}_r + \frac{\omega^2}{c_1^2}\right)}_{\mathcal{L}_1}\Phi = k_z^2\Phi,$$

$$\underbrace{\left(\mathrm{d}_r^2 + r^{-1}\mathrm{d}_r - \frac{1}{r^2} + \frac{\omega^2}{c_2^2}\right)}_{\mathcal{L}_2}\Psi = k_z^2\Psi.$$

Boundary conditions:

$$\sigma_{rr} = \left[-\lambda\left(r^{-2} + \frac{\omega^2}{c_1^2}\right) + 2\mu\partial_r^2\right]\Phi + 2\mu\partial_r\hat{\Psi},$$

$$\hat{\sigma}_{rz} = -2\mu\left(\partial_r^3 + r^{-1}\partial_r^2 - r^{-2}\partial_r + \frac{\omega^2}{c_1^2}\partial_r\right)\Phi + \mu\left(2\partial_r^2 + 2r^{-1}\partial_r - 2r^{-2} + \frac{\omega^2}{c_2^2}\right)\hat{\Psi},$$

where $\hat{\Psi} = ik_z\Psi$ and $\hat{\sigma}_{rz} = ik_z\sigma_{rz}$.

Spectral method

- Differential equations can be very efficiently solved with spectral collocation methods. Orthogonal polynomials of high degree are used as **global** interpolants to approximate the unknown functions of considered differential equations.
- The discrete matrix operator, which approximates the differential operator, is called a **differentiation matrix**.
- The differentiation matrix can be based on **Chebyshev**, Fourier, Hermitian, or other interpolants, which can be differentiated exactly.
- The global interpolant evaluated at N interpolation points is connected to its first derivative by a matrix vector product.
- Evenly spaced points $\rightarrow$ the Runge phenomenon.
- Chebyshev points:

$$x_i = \cos\left(\frac{(i-1)\pi}{N-1}\right), \quad i = 1, \dots, N$$

$$f(x) \approx \sum_{k=1}^{N+1} f(x_k)\, T_k(x)$$

$$f(x_j)^{(\ell)} = \sum_{k=1}^{N+1} \frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}(T_k(x_j)) f(x_k)$$

$$D_{jk}^{(\ell)} = \frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}(T_k(x_j))$$

$$\Downarrow$$

$$\begin{pmatrix} f_1^{(m)} \\ f_2^{(m)} \\ \vdots \\ f_N^{(m)} \end{pmatrix} \approx \begin{pmatrix} D_{11}^{(m)} & D_{12}^{(m)} & \cdots & D_{1N}^{(m)} \\ D_{21}^{(m)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ D_{N1}^{(m)} & \cdots & \cdots & D_{NN}^{(m)} \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}$$

Chebyshev differential matrix, resources

- CHEBFUN (MATLAB toolbox)

  Driscoll, Hale, Trefethen, editors, Chebfun Guide,
  Pafnuty Publications, Oxford, 2014.

- CHEBDIF (MATLAB routine)

  Weideman, Reddy, A MATLAB differentiation matrix suite,
  ACM Trans. Math. Softw. 26, 2000.

- ApproxFun (Julia package)

  URL: github.com/ApproxFun/ApproxFun.jl.git

- pychebfun (Python implementation of chebfun)

  URL: github.com/olivierverdier/pychebfun

Matrix forms

Helmholtz equations:

$$L_1 = D^{(2)} + \mathrm{diag}\left(\frac{1}{r}\right) D^{(1)} + \left(\frac{\omega^2}{c_1^2}\right) \mathbf{I}$$

$$L_2 = D^{(2)} + \mathrm{diag}\left(\frac{1}{r}\right) D^{(1)} - \left(\frac{1}{r^2}\right) \mathbf{I} + \left(\frac{\omega^2}{c_2^2}\right) \mathbf{I}$$

The stress components:

$$S_{r\Phi} = -\lambda \left[\mathrm{diag}\left(\frac{1}{r^2}\right) + \left(\frac{\omega^2}{c_1^2}\right) \mathbf{I}\right] + 2\mu D^{(2)},$$

$$S_{r\hat{\Psi}} = 2\mu D^{(1)},$$

$$S_{z\Phi} = -2\mu \left[D^{(3)} + \mathrm{diag}\left(\frac{1}{r}\right) D^{(2)} - \mathrm{diag}\left(\frac{1}{r^2}\right) D^{(1)} + \left(\frac{\omega^2}{c_1^2}\right) \mathbf{I}\right],$$

$$S_{z\hat{\Psi}} = \mu \left[2D^{(2)} + 2\mathrm{diag}\left(\frac{1}{r}\right) D^{(1)} - \mathrm{diag}\left(\frac{1}{r^2}\right) + \left(\frac{\omega^2}{c_2^2}\right) \mathbf{I}\right].$$

Combination of equations

Helmholtz equations:

$$\underbrace{\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}}_{L} \begin{pmatrix} \Phi \\ \hat{\Psi} \end{pmatrix} = k_z^2 \begin{pmatrix} \Phi \\ \hat{\Psi} \end{pmatrix}$$

The stress components:

$$\begin{pmatrix} \sigma_{rr} \\ \hat{\sigma}_{rz} \end{pmatrix} = \underbrace{\begin{pmatrix} S_{r\Phi} & S_{r\hat{\Psi}} \\ S_{z\Phi} & S_{z\hat{\Psi}} \end{pmatrix}}_{S} \begin{pmatrix} \Phi \\ \hat{\Psi} \end{pmatrix}$$

Embedding boundary conditions

1. The lines in the L matrix corresponding to $r = a$ will be replaced by the corresponding lines of the S matrix.
2. The corresponding values on the right-hand have to be set equal to zero.

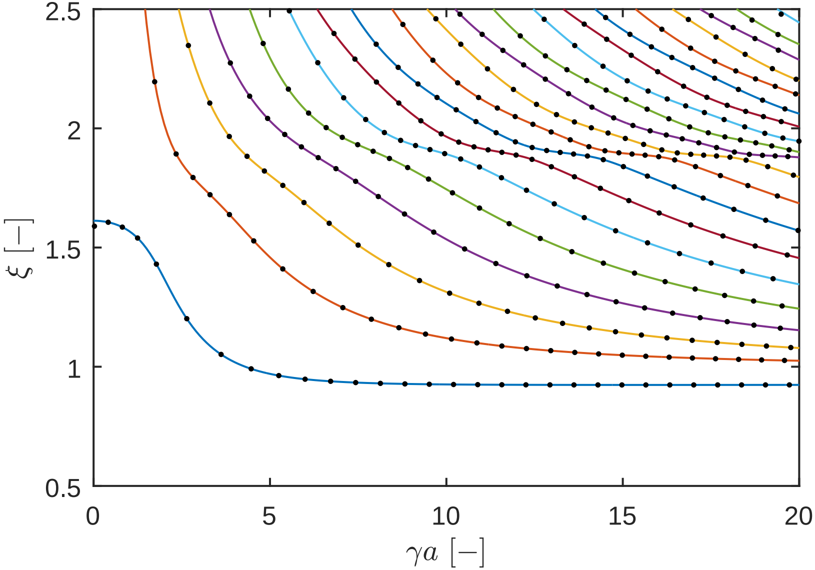
$$\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} \Phi \\ \hat{\Psi} \end{pmatrix} = k_z^2 \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} \Phi \\ \hat{\Psi} \end{pmatrix},$$

where

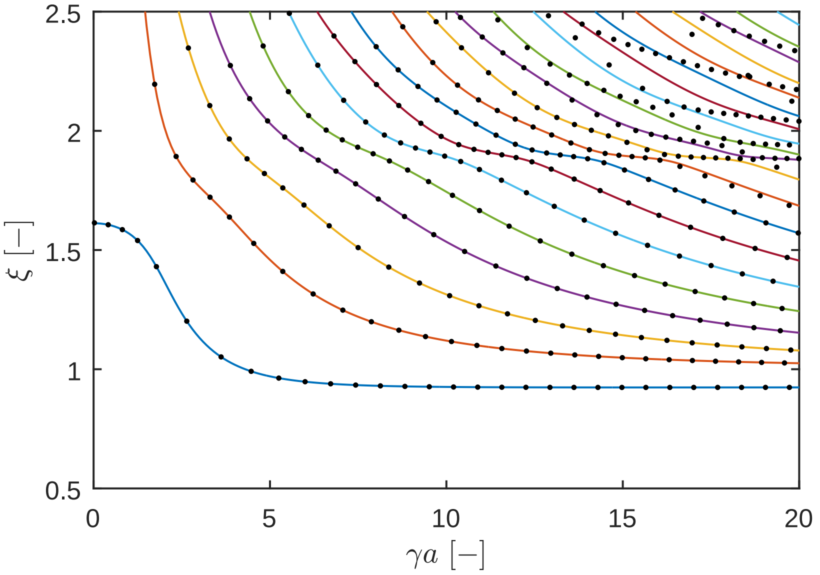
$$M = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix}$$

Generalized eigenvalue problem $\rightarrow$ MATLAB routine `eig`.

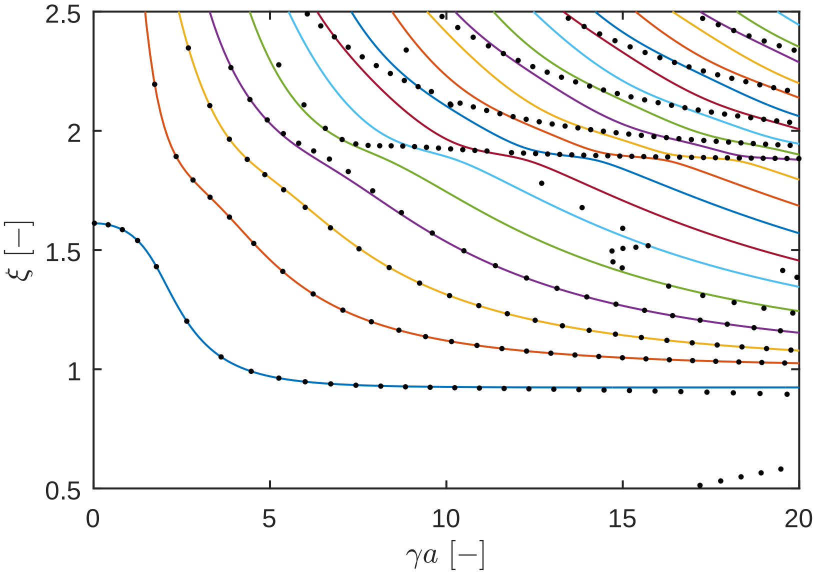
Dispersion curves - spectral method, $N = 30$



Dispersion curves - spectral method, $N = 20$



Dispersion curves - spectral method, $N = 10$



Thank you for your attention!

Any questions?

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