THE USE OF THE CHEBYSHEV INTERPOLATION IN ELASTODYNAMICS PROBLEMS

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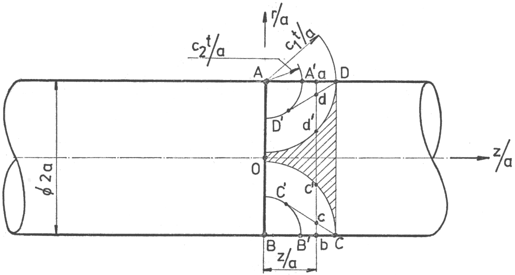
Motivation



Valeš, F., Podélný ráz polonekonečných válcových elastických tyčí kruhového průřezu, part I (Z681/79) and II (Z839/83), ÚT AVČR Praha, (in Czech).

$$\int_0^\infty \frac{(2-\xi^2)J_1(r\,\gamma a\,B)J_1(\gamma a\,A) - 2J_1(\gamma a\,B)J_1(r\,\gamma a\,A)}{\gamma a\,\xi\,B\,M}\cos(z\,\gamma a)\sin(t\,\xi\,\gamma a)\,d\gamma a,$$

where

$$\begin{aligned} M = \; & -\gamma a^2\,((2-\xi^2)^2/A + 4\,\kappa\,A)\,J_0(\gamma a\,B)\,J_0(\gamma a\,A) \\ & +\gamma a\,((2-\xi^2)\,(4+(2-\xi^2)/(\xi^2-1)) + 2\,\kappa\,\xi^2)\,J_0(\gamma a\,B)\,J_1(\gamma a\,A) \\ & -2\,\gamma a\,(2-\xi^2)\,B/A\,J_1(\gamma a\,B)\,J_0(\gamma a\,A) \\ & +(2\,B\,(2\,\gamma a^2 - (2-\xi^2)/(\xi^2-1)) + \kappa\,\gamma a^2\,(2-\xi^2)^2/B)\,J_1(\gamma a\,B)\,J_1(\gamma a\,A), \end{aligned}$$

- a radius of the semi-infinite bar,
- γ wavenumber,
- ξ the ratio of the phase velocity and the shear wave velocity, $\xi(\gamma a)$,
- κ the ratio of the squares of the phase velocities,
- A $\sqrt{\kappa\xi^2-1}$,
- B $\sqrt{\xi^2-1}$,
- J the Bessel function of the first kind.

Dispersion curves ($\xi - \gamma a$)

The dispersion relations $f(\xi, \gamma a)$ is defined as

$$\left(2-\xi^2\right)^2 J_0(\gamma a A)\, J_1(\gamma a B) + 4ABJ_1(\gamma a A)\, J_0(\gamma a B) - \frac{2\xi^2}{\gamma a} A J_1(\gamma a A)\, J_1(\gamma a B) = 0,$$

where

- a radius of the semi-infinite bar,
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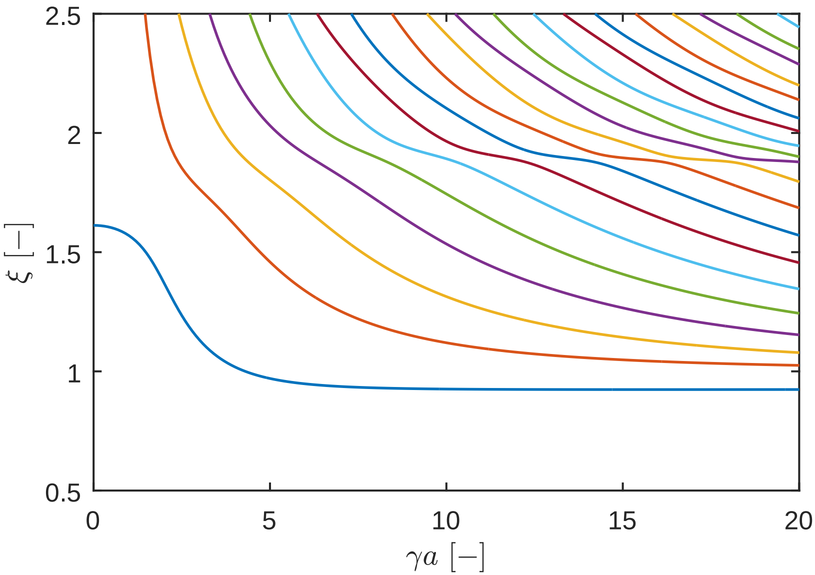
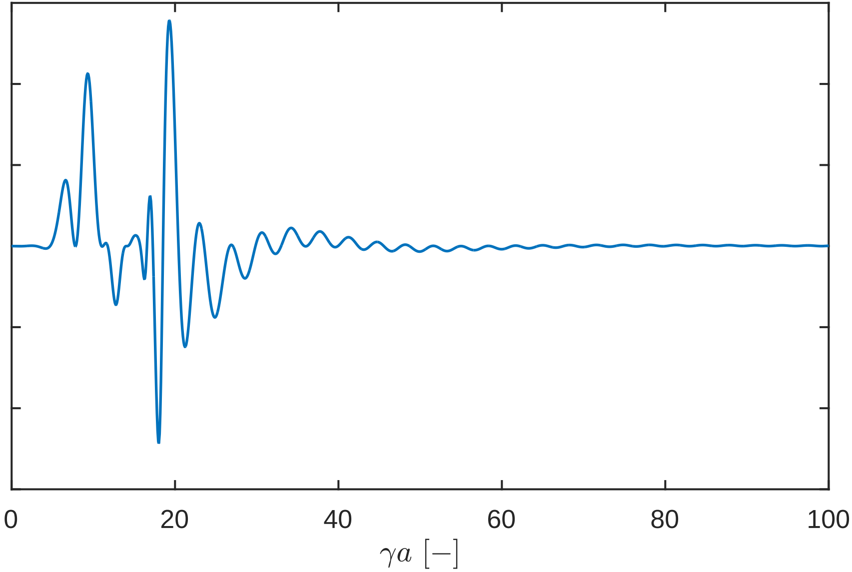


$\mu = 0.3, \mathrm{dc} = 10, \mathrm{r} = 0.5, z = 1, \mathrm{t} = 1$



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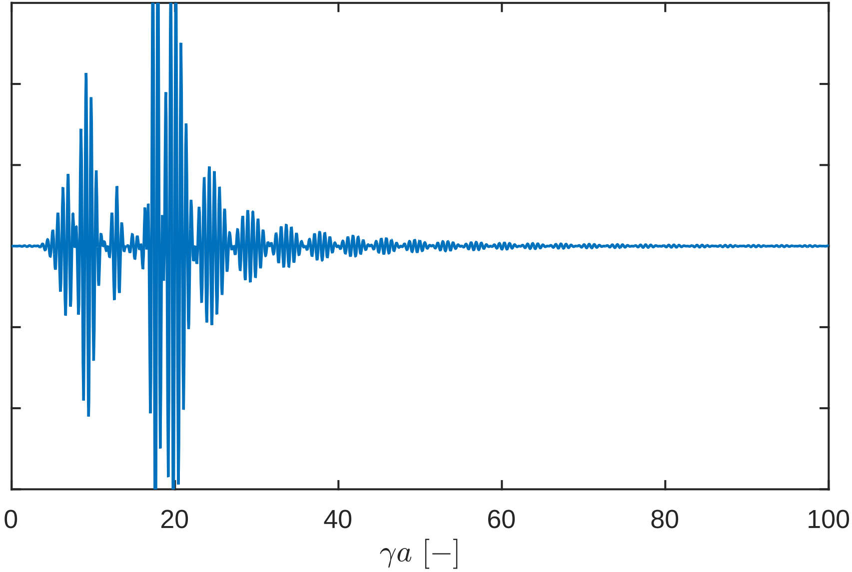

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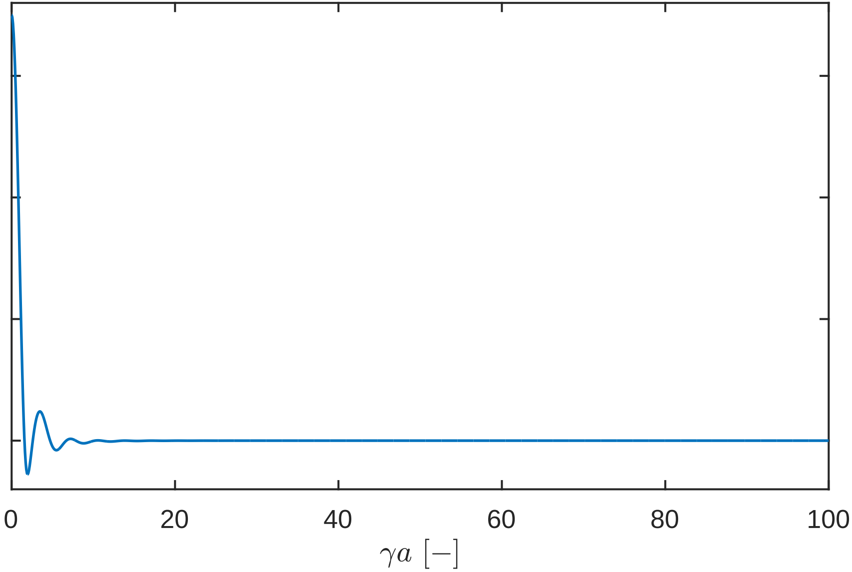


Current calculation method

- For the purpose of the speed up calculation, the dispersion curves are precalculated in equidistant points of γa
  For example: $\gamma a_{min} = 0.001$, $\gamma a_{max} = 500$, $\Delta \gamma a = 0.001$.
- Method for numerical integration: Simpson's rule (points are equally spaced).

The disadvantages of this process are:

1. impossibility to integrate from zero,
2. problematic choice of $\Delta \gamma a$.

The proposed methodology for calculating

- In order to remove the first restriction,
  another form of the dispersion relations is used.
- In order to remove the second restriction,
  the integration method with unequally spaced points is used.
  For non-periodic functions, methods with unequally spaced points such as *Gaussian quadrature* and *Clenshaw–Curtis quadrature* are generally far more accurate.
  For using of these integration methods, the dispersion curves were approximated by **the Chebyshev polynomials**.

Dispersion curves ($\zeta - \gamma a$)

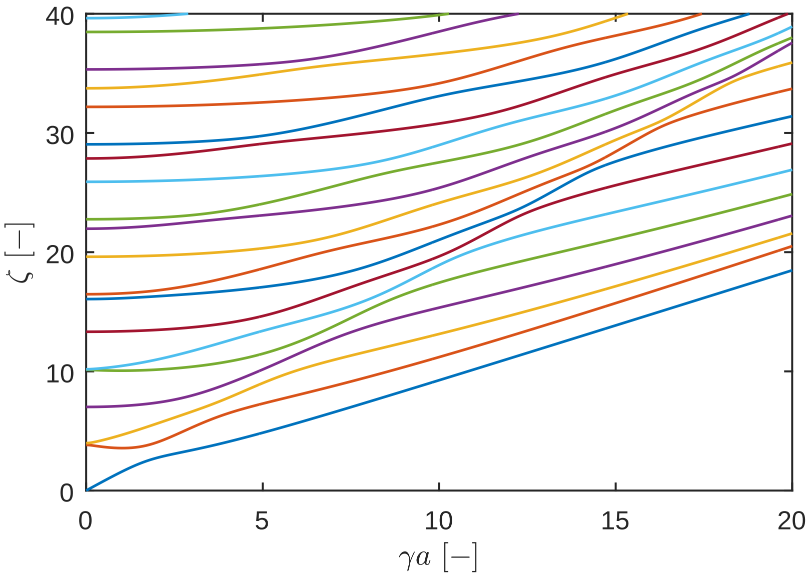
After making the substitution, $\zeta = \xi \gamma a$, the dispersion relation $g(\zeta, \gamma a)$ is defined as

$$\left(2\gamma a^2 - \zeta^2\right)^2 J_0\left(C\right) J_1\left(D\right) + 4\gamma a^2 C D J_1\left(C\right) J_0\left(D\right) - 2\zeta^2 C J_1\left(C\right) J_1\left(D\right) = 0,$$

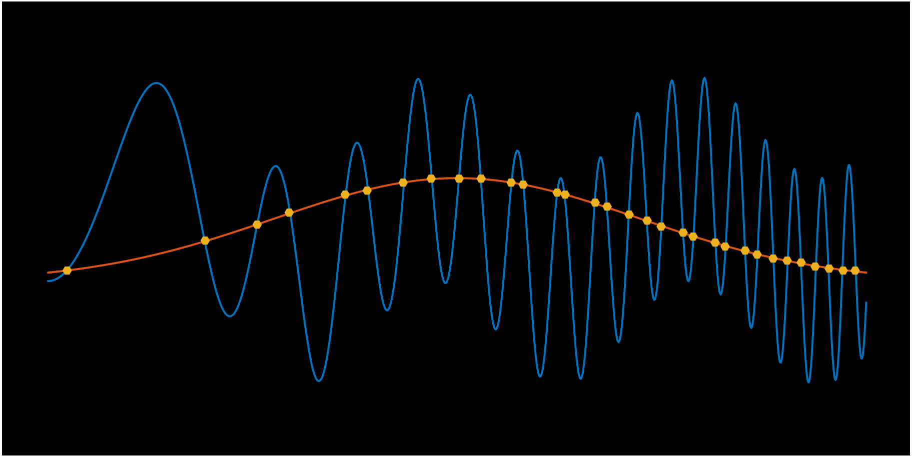
where

- a radius of the semi-infinite bar,
- γ wavenumber,
- ζ normalized angular frequency, $\xi\gamma a$,
- κ the ratio of the squares of the phase velocities,
- C $\sqrt{\kappa\zeta^2 - \gamma a^2}$,
- D $\sqrt{\zeta^2 - \gamma a^2}$,
- J the Bessel function of the first kind.

Dispersion curves ($\zeta - \gamma a$)



Chebyshev interpolation

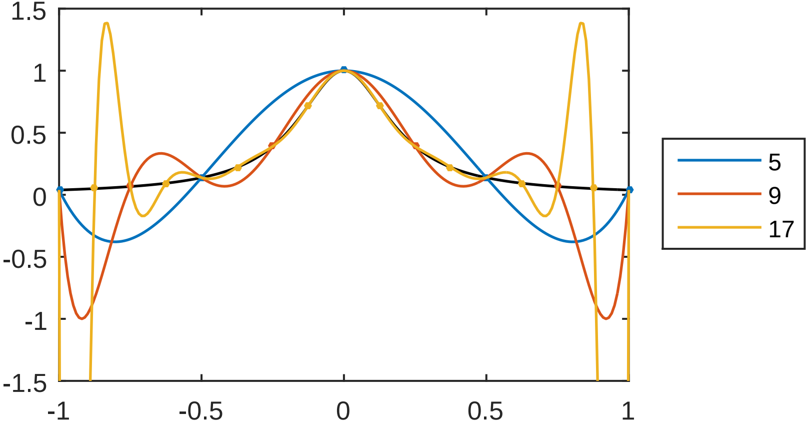


Chebyshev interpolation

```
% Define two functions
f = chebfun(@(x) sin(x.^2)+sin(x).^2, [0,10]);
g = chebfun(@(x) exp(-(x-5).^2/10), [0,10]);

% Compute their intersections
rr = roots(f - g);

% Plot the functions
plot([f g]), hold on

% Plot the intersections
plot(rr, f(rr), 'o')
```

The Runge's phenomenon

- Problem of oscillation at the edges of an interval that occurs when using polynomial interpolation with polynomials of high degree over a set of equispaced interpolation points.



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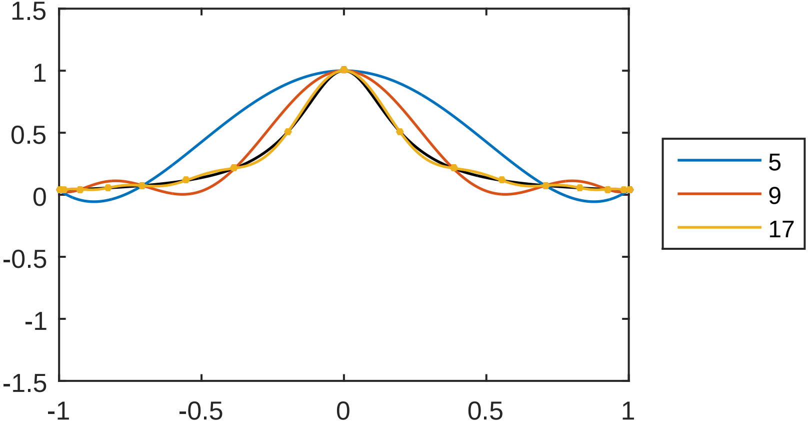


Chebyshev polynomials of the first kind, I

- The recurrence relation

$$
\begin{aligned}
T_0(x) &= 1, \\
T_1(x) &= x, \\
T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x).
\end{aligned}
$$

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1
\end{aligned}
$$

Chebyshev polynomials of the first kind, II

- Trigonometric definition

$$T_n(x) = \cos(n \arccos x) = \cosh(n \text{ arcosh } x)$$

- Roots
  n different simple roots, called Chebyshev roots, in the interval $[-1, 1]$.

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, \ldots, n$$

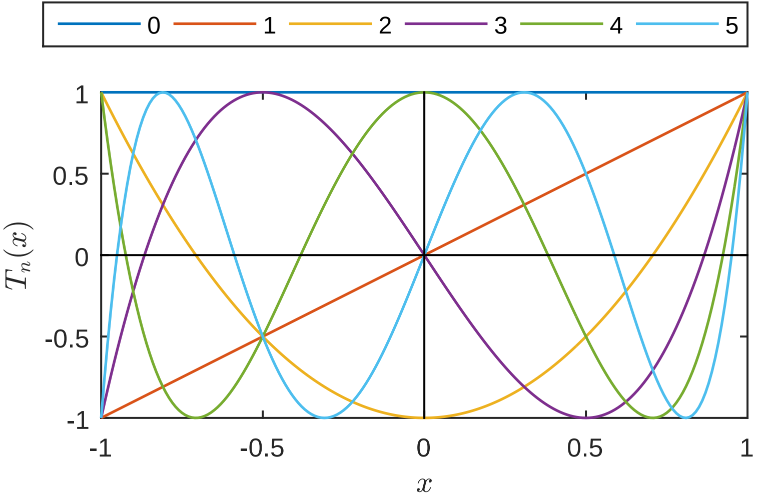
- Extrema

$$x_k = \cos\left(\tfrac{k}{n}\pi\right), \quad k = 0, \ldots, n$$

  All of the extrema have values that are either -1 or 1.
  Extrema at the endpoints, given by:

$$\begin{aligned} T_n(1) &= 1 \\ T_n(-1) &= (-1)^n \end{aligned}$$

Chebyshev polynomials of the first kind, III



Chebyshev interpolation

- Basic idea:
  Represent functions using interpolants through (suitably rescaled) Chebyshev nodes
  $$x_j = -\cos\left(\frac{j\pi}{n}\right), \qquad 0 \leqslant j \leqslant n.$$
- Such interpolants have excellent approximation properties.
- Interpolants are constructed adaptively, more and more points used, until coefficients in Chebyshev series fall below machine precision.

Chebyshev interpolation, Algorithm I

1. Choose the following:
   1.1 γa
   1.2 Search interval, $x \in [a, b]$.
   The search interval must be chosen by physical and mathematical analysis of the individual problem. The choice of the search interval $[a, b]$ depends on the user's knowledge of the physics of his/her problem, and no general rules are possible.
   1.3 The number of grid points, N.
   N may be chosen by setting $N = 1 + 2^m$ and the increasing N until the Chebyshev series displays satisfactory convergence. To determine when N is sufficiently high, we can examine the Chebyshev coefficients a_j, which decrease exponentially fast with j.
2. Compute a Chebyshev series, including terms up to and including T_N, on the interval $x \in [a, b]$.
   2.1 Create the interpolation points (Lobatto grid):

$$x_k \equiv \frac{b-a}{2} \cos\left(\pi \frac{k}{N}\right) + \frac{b+a}{2}, \quad k = 0, 1, 2, \ldots, N.$$

Chebyshev interpolation, Algorithm II

2.2 Compute the elements of the $(N+1) \times (N+1)$ interpolation matrix. Define $p_j = 2$ if $j = 0$ or $j = N$ and $p_j = 1, j \in [1, N-1]$. Then the elements of the interpolation matrix are

$$I_{jk} = \frac{2}{p_j p_k N} \cos\left(j\pi \frac{k}{N}\right).$$

2.3 Compute the grid-point values of $f(x)$, the function to be approximated:

$$f_k \equiv f(x_k), \quad k = 0, 1, \ldots, N.$$

2.4 Compute the coefficients through a vector-matrix multiply:

$$a_j = \sum_{k=0}^{N} I_{jk} f_k, \quad j = 0, 1, 2, \ldots..., N.$$

The approximation is

$$f_N \approx \sum_{j=0}^{N} a_j T_j \left(\frac{2x - (b+a)}{b-a}\right) = \sum_{j=0}^{N} a_j \cos\left\{j \arccos\left(\frac{2x - (b+a)}{b-a}\right)\right\}.$$

Chebyshev interpolation, resources

- CHEBFUN (MATLAB toolbox)

  Driscoll, T.A., Hale, N., Trefethen, L.N., editors, Chebfun Guide, Pafnuty Publications, Oxford, 2014.

- ApproxFun (Julia package)

  URL: github.com/ApproxFun/ApproxFun.jl.git

- pychebfun (Python implementation of chebfun)

  URL: github.com/olivierverdier/pychebfun

The Chebyshev approximation properties for any dispersion curves ($\zeta - \gamma\alpha$)

Curve #	Vertical scale	Length (splitting=off)	Length (splitting=on)
1	93	396	168
2	100	454	218
3	100	455	209
10	100	1821	301
20	120	4226	597
30	140	6527	729

Compression: 100 curves for $\gamma\alpha$ from 0.001 to 100 with step 0.001 take the 80MB file.
100 curves for $\gamma\alpha$ from $\mathbf{0}$ to 100 created by the Chebyshev approximation take only the 1.3MB file.

Speed-up: While using the *Gaussian quadrature*, the calculation time has provided a 100x speedup over the original.

Thank you for your attention!

Any questions?

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