THE ROOT-FINDING OF DISPERSION CURVES IN A BAR IMPACT PROBLEM

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Motivation: Bar impact problem

To calculate the stress wave propagation in a bar impact problem
it is used the integration along the dispersion curves.

This dispersion relations $f(x, \gamma a)$ is defined as

$$\left(2 - x^2\right)^2 J_0(\gamma a A)\, J_1(\gamma a B) + 4AB J_1(\gamma a A)\, J_0(\gamma a B) - \frac{2x^2}{\gamma a} A J_1(\gamma a A)\, J_1(\gamma a B) = 0,$$

where

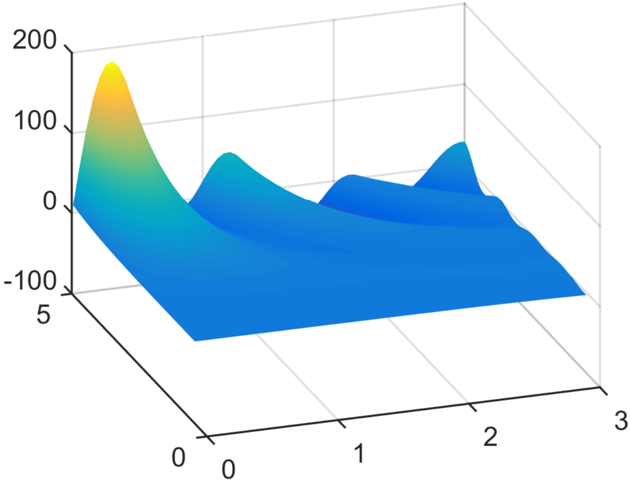
- a radius of the semi-infinite bar,
- γ wavenumber,
- x the ratio of the phase velocity and the shear wave velocity,
- κ the ratio of the squares of the phase velocities,
- A $\sqrt{\kappa x^2 - 1}$,
- B $\sqrt{x^2 - 1}$,
- J the Bessel function of the first kind.

Transcendental equations

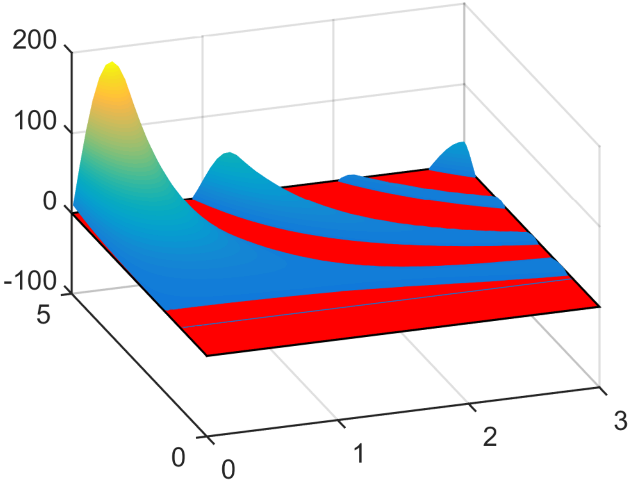
- A transcendental equation is an equation containing a transcendental function of the variable(s) being solved for.
- The most familiar transcendental functions are:
  - the logarithm,
  - the exponential (with any non-trivial base),
  - the trigonometric,
  - the hyperbolic functions,
  - and the inverses of all of these.
- Less familiar are:
  - The special functions of analysis, such as the gamma, elliptic, and zeta functions, all of which are transcendental.
  - The generalized hypergeometric and Bessel functions are transcendental in general, but algebraic for some special parameter values.

WIKIPEDIA, Transcendental function

$f(x, \gamma a)$



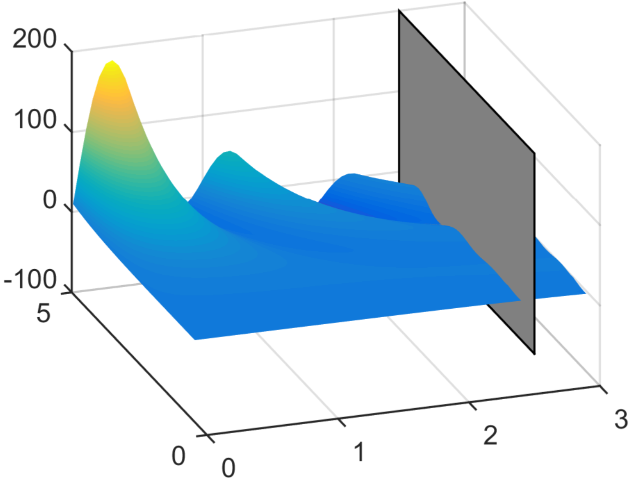
Flooded $f(x, \gamma a)$



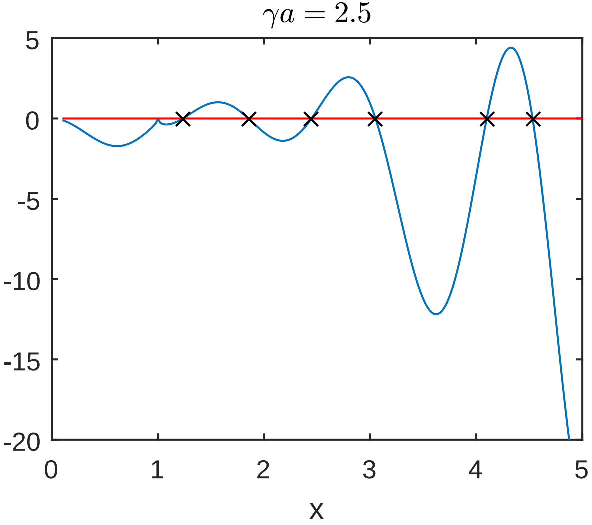
Solution methods

1. Root-finding
2. Interval arithmetics
3. Chebyshev interpolation
4. Marching squares
5. Marching triangles
6. ...

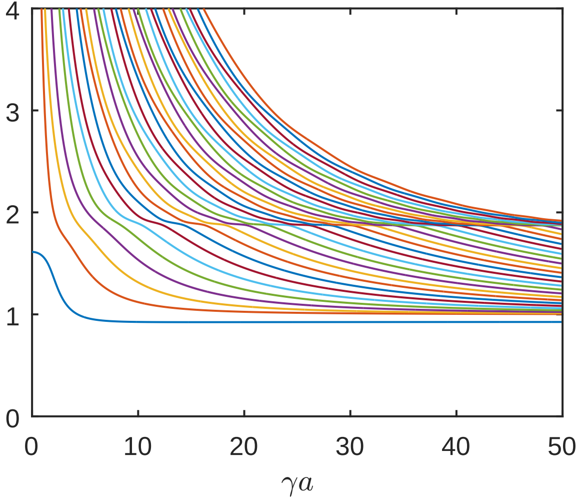
Root-finding, 3D



Root-finding, cut



Root-finding, DC



Root-finding, limits

1. $0 < x < 1$
$$\lim_{\gamma a \to +\infty} x = 0.92741271$$
2. $1 < x < 1/\sqrt{\kappa}$
$$\lim_{\gamma a \to 0_+} x = 1.61245155$$
3. $x > 1/\sqrt{\kappa}$

$$\lim_{\gamma a \to 0_+} x = \begin{cases} J_1(\gamma a\, x) = 0, \\ \left[\gamma a\, x\, J_0\left(\sqrt{\kappa}\,\gamma a\, x\right) - 2\sqrt{\kappa} J_1\left(\sqrt{\kappa}\,\gamma a\, x\right)\right] = 0. \end{cases}$$

Root-finding, Algorithm

- Stepping in γa,
- The guess for the first steps by means of the limit,
- The guess for the next steps by means of the extrapolation,
- Newton method,
- Parallel processing.

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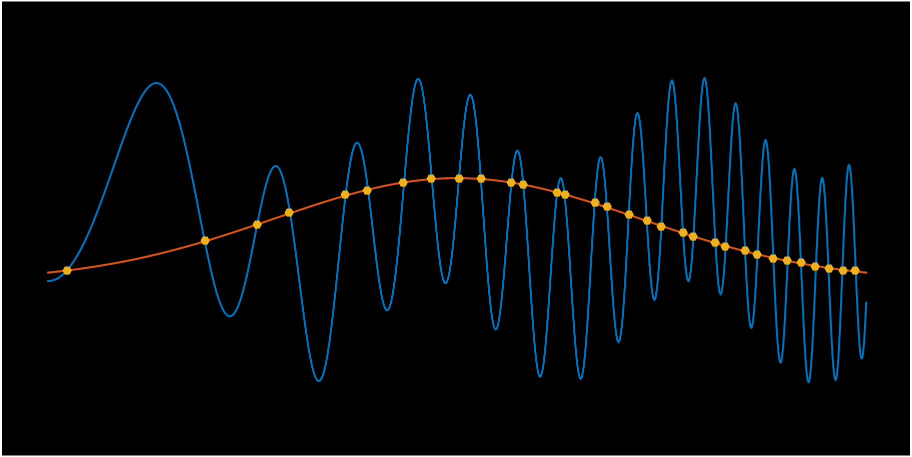
Disadvantages

- Root skip,
- Lazy (particularly for random γa),
- Need of differentiation (Newton method).

Chebyshev interpolation



Chebyshev interpolation

```
% Define two functions
f = chebfun(@(x) sin(x.^2)+sin(x).^2, [0,10]);
g = chebfun(@(x) exp(-(x-5).^2/10), [0,10]);

% Compute their intersections
rr = roots(f - g);

% Plot the functions
plot([f g]), hold on

% Plot the intersections
plot(rr, f(rr), 'o')
```

Chebyshev interpolation

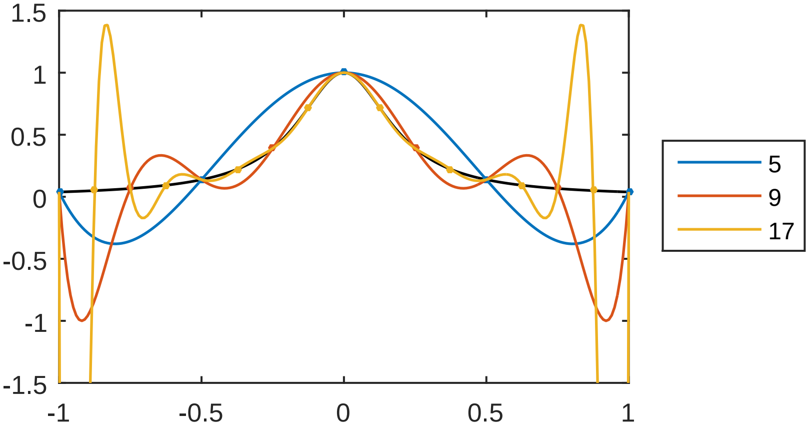
Solve-the-proxy methods in one unknown:

Approximation	Name
Piecewise linear interpolation	Make-a-Graph-Stupid algorithm
Linear Taylor series	Newton–Raphson iteration
Linear interpolant	secant iteration
Quadratic Taylor series	Cauchy's method
Quadratic interpolant	Muller's method
Inverse quadratic interpolant	Brent's algorithm
(Linear-over-linear) Padé approximant	Halley's scheme
(Quadratic-over-quadratic) Padé approximant	Shafer's method
Chebyshev polynomial interpolant	**Chebyshev proxy method**

J.P. Boyd: Finding the Zeros of a Univariate Equation, SIAM Rev., 55(2)

The Runge's phenomenon

- Problem of oscillation at the edges of an interval that occurs when using polynomial interpolation with polynomials of high degree over a set of equispaced interpolation points.



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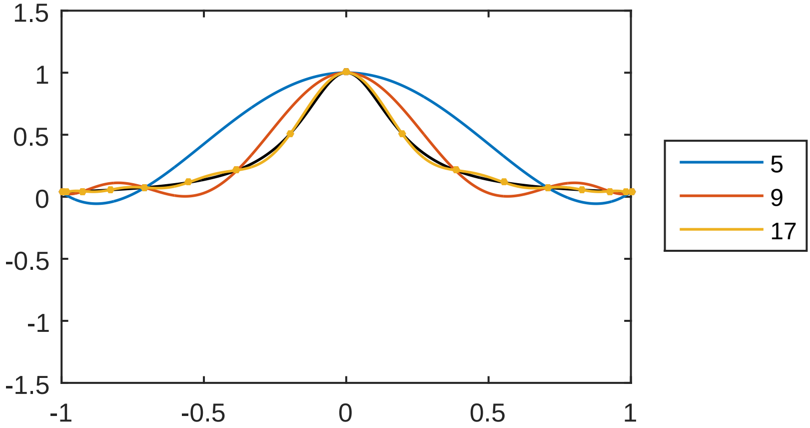


Chebyshev polynomials of the first kind, I

- The recurrence relation

$$
\begin{aligned}
T_0(x) &= 1, \\
T_1(x) &= x, \\
T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x).
\end{aligned}
$$

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1
\end{aligned}
$$

Chebyshev polynomials of the first kind, II

- Trigonometric definition

$$T_n(x) = \cos(n \arccos x) = \cosh(n \operatorname{arcosh} x)$$

- Roots
  n different simple roots, called Chebyshev roots, in the interval $[-1, 1]$.

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, \dots, n$$

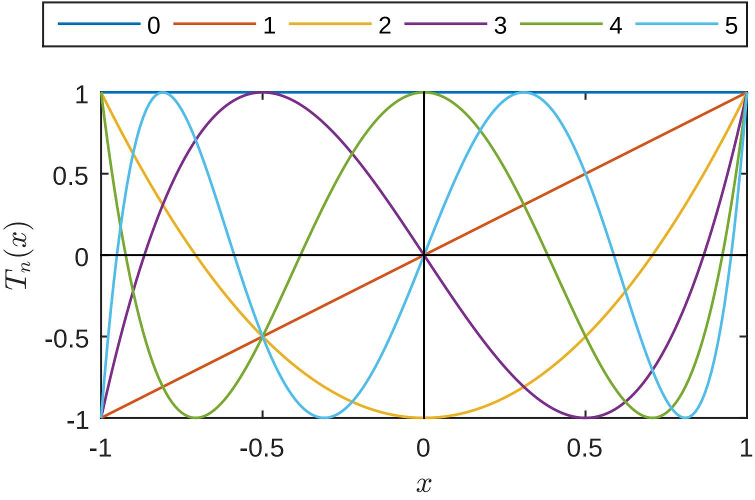
- Extrema

$$x_k = \cos\left(\tfrac{k}{n}\pi\right), \quad k = 0, \dots, n$$

  All of the extrema have values that are either -1 or 1.
  Extrema at the endpoints, given by:

$$\begin{aligned} T_n(1) &= 1 \\ T_n(-1) &= (-1)^n \end{aligned}$$

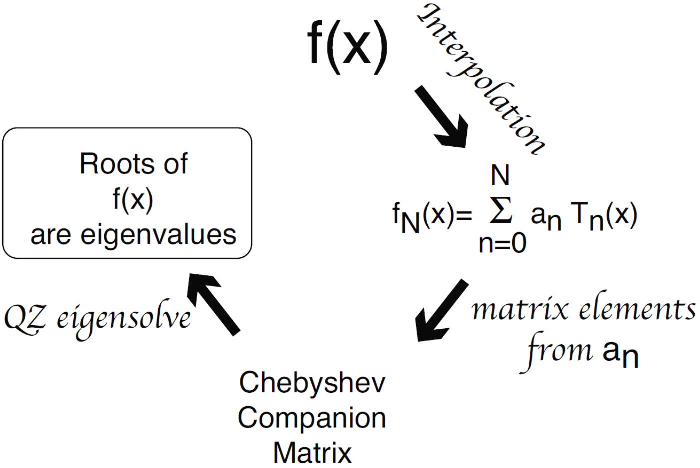
Chebyshev polynomials of the first kind, III



Chebyshev interpolation

- Basic idea:
  Represent functions using interpolants through (suitably rescaled) Chebyshev nodes
$$x_j = -\cos\left(\frac{j\pi}{n}\right), \qquad 0 \leqslant j \leqslant n.$$
- Such interpolants have excellent approximation properties.
- Interpolants are constructed adaptively, more and more points used, until coefficients in Chebyshev series fall below machine precision.

Chebyshev-proxy rootfinder



Chebyshev interpolation, Algorithm I

1. Choose the following:
   1.1 γa
   1.2 Search interval, $x \in [a, b]$.
   The search interval must be chosen by physical and mathematical analysis of the individual problem. The choice of the search interval $[a, b]$ depends on the user's knowledge of the physics of his/her problem, and no general rules are possible.
   1.3 The number of grid points, N.
   N may be chosen by setting $N = 1 + 2^m$ and the increasing N until the Chebyshev series displays satisfactory convergence. To determine when N is sufficiently high, we can examine the Chebyshev coefficients a_j, which decrease exponentially fast with j.
2. Compute a Chebyshev series, including terms up to and including T_N, on the interval $x \in [a, b]$.
   2.1 Create the interpolation points (Lobatto grid):

$$x_k \equiv \frac{b-a}{2} \cos\left(\pi \frac{k}{N}\right) + \frac{b+a}{2}, \quad k = 0, 1, 2, \ldots, N.$$

Chebyshev interpolation, Algorithm II

2.2 Compute the elements of the $(N+1) \times (N+1)$ interpolation matrix. Define $p_j = 2$ if $j = 0$ or $j = N$ and $p_j = 1, j \in [1, N-1]$. Then the elements of the interpolation matrix are

$$I_{jk} = \frac{2}{p_j p_k N} \cos\left(j\pi \frac{k}{N}\right).$$

2.3 Compute the grid-point values of $f(x)$, the function to be approximated:

$$f_k \equiv f(x_k), \quad k = 0, 1, \ldots, N.$$

2.4 Compute the coefficients through a vector-matrix multiply:

$$a_j = \sum_{k=0}^{N} I_{jk} f_k, \quad j = 0, 1, 2, \ldots \ldots, N.$$

The approximation is

$$f_N \approx \sum_{j=0}^{N} a_j T_j \left(\frac{2x - (b+a)}{b-a}\right) = \sum_{j=0}^{N} a_j \cos\left\{j \arccos\left(\frac{2x - (b+a)}{b-a}\right)\right\}.$$

Chebyshev interpolation, Algorithm III

3. Compute the roots of f_N as eigenvalues of the Chebyshev–Frobenius matrix. Frobenius showed that the roots of a polynomial in monomial form are also the eigenvalues of the matrix which is now called the *Frobenius companion matrix*. Day and Romero developed a general formalism for deriving the *Frobenius matrix* for any set of orthogonal polynomials.
4. Refine the roots by a Newton iteration with $f(x)$ itself. Once a good approximation to a root is known, it is common to *polish* the root to close to machine precision by one or two Newton iterations.

Chebyshev interpolation, Numerical experiments

- ApproxFun (Julia package)

  URL: github.com/ApproxFun/ApproxFun.jl.git

- CHEBFUN (MATLAB toolbox)

  Driscoll, T.A., Hale, N., Trefethen, L.N., editors, Chebfun Guide, Pafnuty Publications, Oxford, 2014.

Thank you for your attention!

Any questions?

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