COMPARISON OF THREE ANALYTICAL METHODS BASED ON THREE BASIC TASKS OF ELASTODYNAMICS

Petr Hora

Institute of Thermomechanics AS CR, v. v. i.
Prague, Czech Republic

COMPUTATIONAL MECHANICS 2013

November 4 – 6, 2013
Špičák, Czech Republic

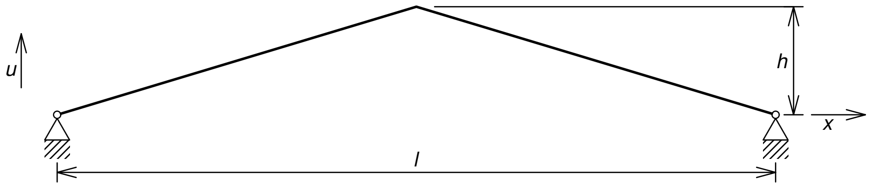
Introduction

The main aim of the contribution is to illustrate:

- Three various methods of solutions:
  1. the expansion in a series of normal functions,
  2. the numerical inverse Laplace transform
  3. a generalized ray theory.
- On three tasks of the nonstationary state of stress:
  1. uniform string, plucked aside at its centre,
  2. torsionally loaded disc with concentric hole,
  3. point loaded thick plate.

The advantages and drawbacks of these methods will be shown.

Problem of a uniform string, plucked aside at its centre



where

l - length of string,

h - initial displacement of string.

Strutt, J. W. (Baron Rayleigh), The Theory of Sound, Volume I, Macmillan and Co., London, 1877.

WRONG

RIGHT

- Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where

c - velocity of transverse vibrations.

- Initial conditions
- Boundary conditions
- The Laplace transform of transverse vibrations u

$$\bar{u}(p, x) = \frac{kx}{p} - \frac{kc}{p^2} \frac{\sinh \frac{px}{c}}{\cosh \frac{pl}{2c}},$$

where

p - parameter of Laplace transform,
k - $2h/l$.

- Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where

c - velocity of transverse vibrations.

- Initial conditions
- Boundary conditions
- The Laplace transform of transverse vibrations u

$$\bar{u}(p,x) = \frac{kx}{p} - \frac{kc}{p^2} \frac{\sinh \frac{px}{c}}{\cosh \frac{pl}{2c}},$$

where

p - parameter of Laplace transform,
k - $2h/l$.

INVERSION?

1 - The expansion of u in a series of normal functions.

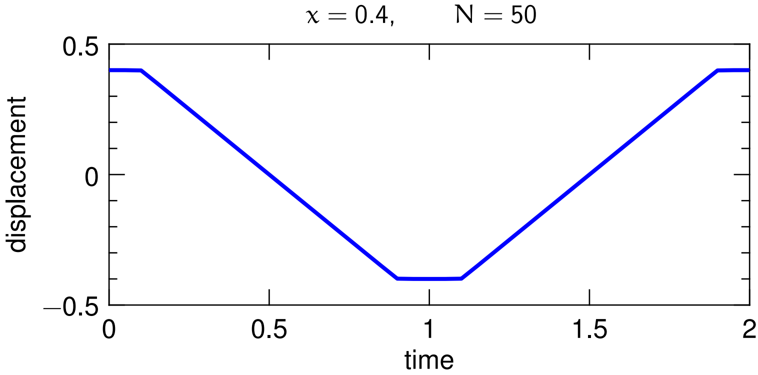
- a sum of residues

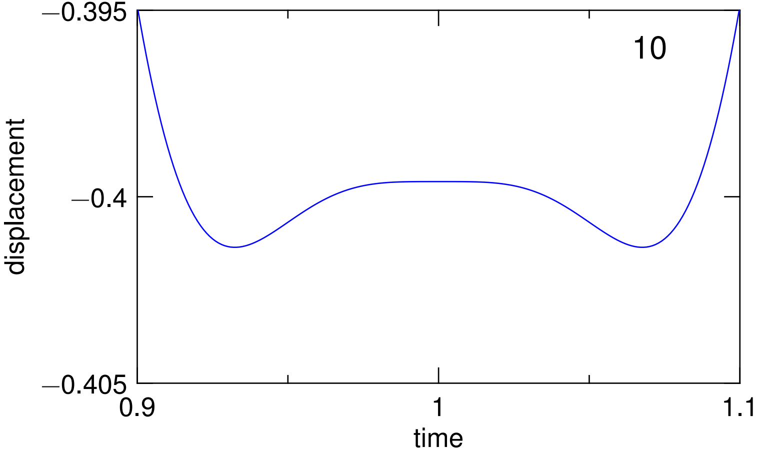
$$u(t,x) = \frac{4kl}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin\left\{(2n+1)\frac{\pi}{l}x\right\} \cos\left\{(2n+1)\frac{\pi c}{l}t\right\}$$

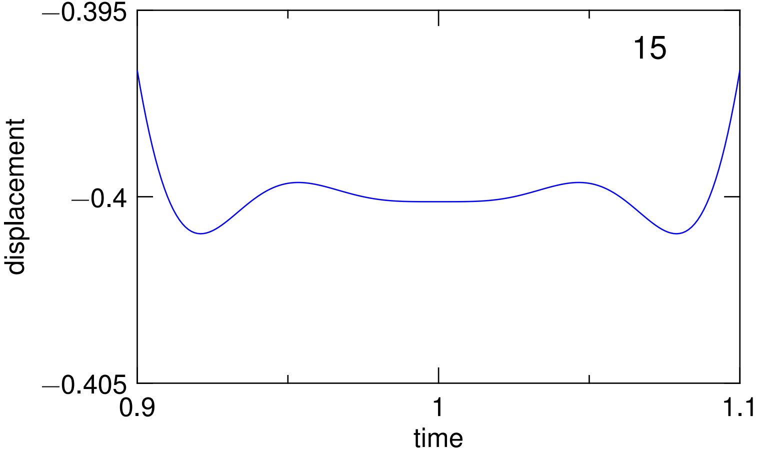
1 - The expansion of u in a series of normal functions.

- a sum of residues

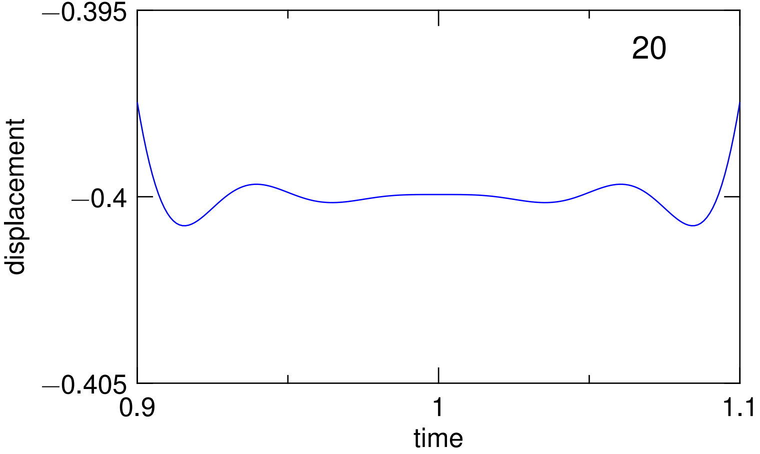
$$u(t,x) = \frac{4kl}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin\left\{(2n+1)\frac{\pi}{l}x\right\} \cos\left\{(2n+1)\frac{\pi c}{l}t\right\}$$



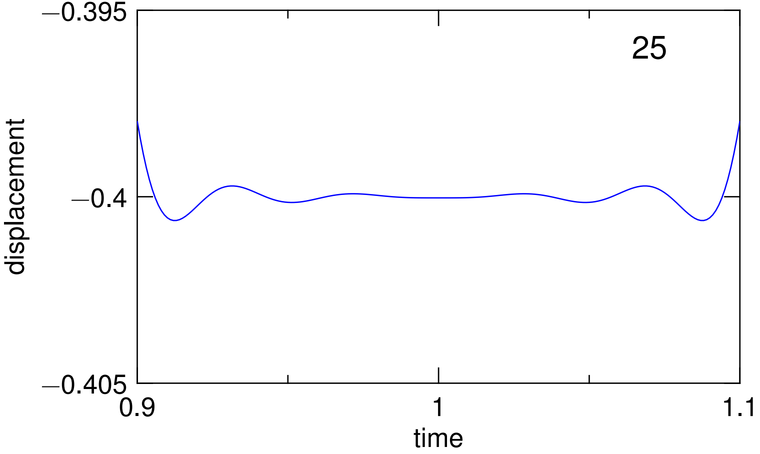

−0.395
−0.4
−0.405
0.9
1
1.1
10
displacement
time



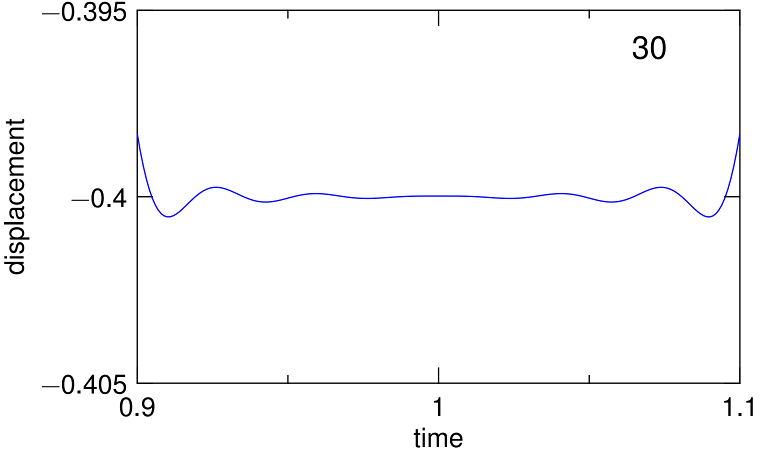
15
−0.395
−0.4
−0.405
displacement
0.9
1
1.1
time



−0.395
−0.4
−0.405
0.9
1
1.1
20
displacement
time



−0.395
−0.4
−0.405
displacement
25
0.9
1
1.1
time



30
−0.395
−0.4
−0.405
displacement
0.9
1
1.1
time

2 - Bromwich's expansion

Bromwich, T.J.I'A.,
Normal Coordinates in Dynamical Systems,
Proc. London. Mat. Soc. (15) (1916) 401-448.

2 - Bromwich's expansion

Bromwich, T.J.I'A.,
Normal Coordinates in Dynamical Systems,
Proc. London. Mat. Soc. (15) (1916) 401-448.

It is often possible to replace the expression for $\bar{u}$ by a series of terms such as e^{-ap}/p^{n+1}, where a is real and n is an integer.

2 - Bromwich's expansion

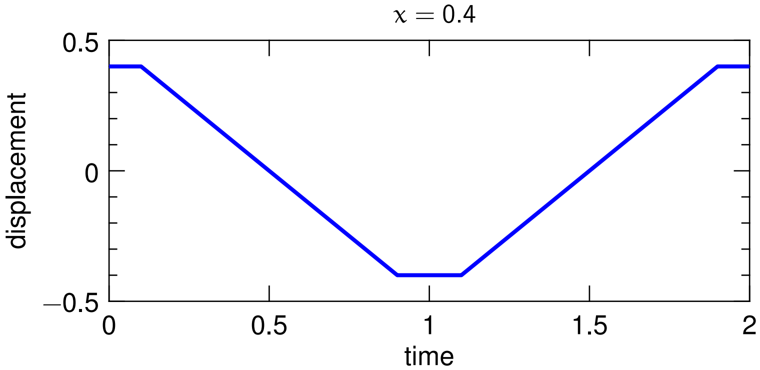
Bromwich, T.J.I'A.,
Normal Coordinates in Dynamical Systems,
Proc. London. Mat. Soc. (15) (1916) 401-448.

It is often possible to replace the expression for $\bar{u}$ by a series of terms such as e^{-ap}/p^{n+1}, where a is real and n is an integer.

$$\begin{aligned}
\bar{u}(p,x) &= \frac{kx}{p} - \frac{kc}{p^2}\frac{\sinh\frac{px}{c}}{\cosh\frac{pl}{2c}} \\
&= \frac{kx}{p} - \frac{kc}{p^2}\left[e^{-\frac{p}{c}(\frac{l}{2}-x)} - e^{-\frac{p}{c}(\frac{l}{2}+x)}\right]\frac{1}{1+e^{-\frac{p}{c}l}} \\
&= \frac{kx}{p} - \frac{kc}{p^2}\left[e^{-\frac{p}{c}(\frac{l}{2}-x)} - e^{-\frac{p}{c}(\frac{l}{2}+x)}\right]\left[1 - e^{-\frac{p}{c}l} + e^{-2\frac{p}{c}l} - \dots\right]
\end{aligned}$$

$$
u(t,x) = \begin{cases} kx & 0 < t < (\frac{1}{2}l - x)/c, \\ k(\frac{1}{2}l - ct) & (\frac{1}{2}l - x)/c < t < (\frac{1}{2}l + x)/c, \\ -kx & (\frac{1}{2}l + x)/c < t < (\frac{3}{2}l - x)/c, \\ k(ct - \frac{3}{2}l) & (\frac{3}{2}l - x)/c < t < (\frac{3}{2}l + x)/c, \\ kx & (\frac{3}{2}l + x)/c < t < (\frac{5}{2}l - x)/c, \\ \text{and so on.} \end{cases}
$$

$$u(t,x) = \begin{cases} kx & 0 < t < (\frac{1}{2}l - x)/c, \\ k(\frac{1}{2}l - ct) & (\frac{1}{2}l - x)/c < t < (\frac{1}{2}l + x)/c, \\ -kx & (\frac{1}{2}l + x)/c < t < (\frac{3}{2}l - x)/c, \\ k(ct - \frac{3}{2}l) & (\frac{3}{2}l - x)/c < t < (\frac{3}{2}l + x)/c, \\ kx & (\frac{3}{2}l + x)/c < t < (\frac{5}{2}l - x)/c, \\ \text{and so on.} & \end{cases}$$



3 - The numerical inverse Laplace transform

3 - The numerical inverse Laplace transform

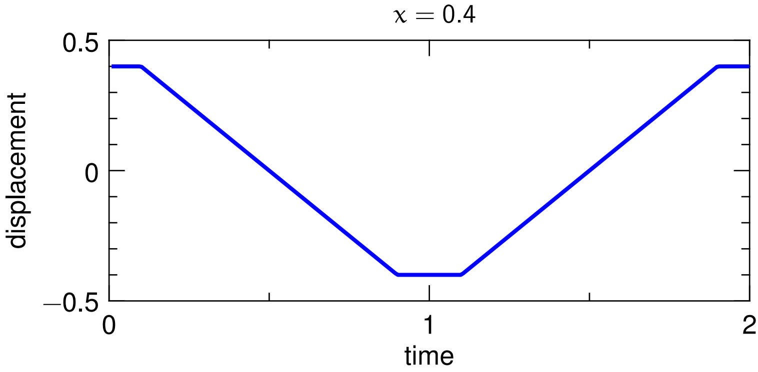
- The FFT based NILT,
- ϵ-algorithm for accelerating convergence of the residual infinite series.

Brančík, L., Programs for fast numerical inversion of Laplace transforms in Matlab language environment, Proceedings of 7th Conference MATLAB'99, Prague, Czech Republic, 1999, pp. 27-39.

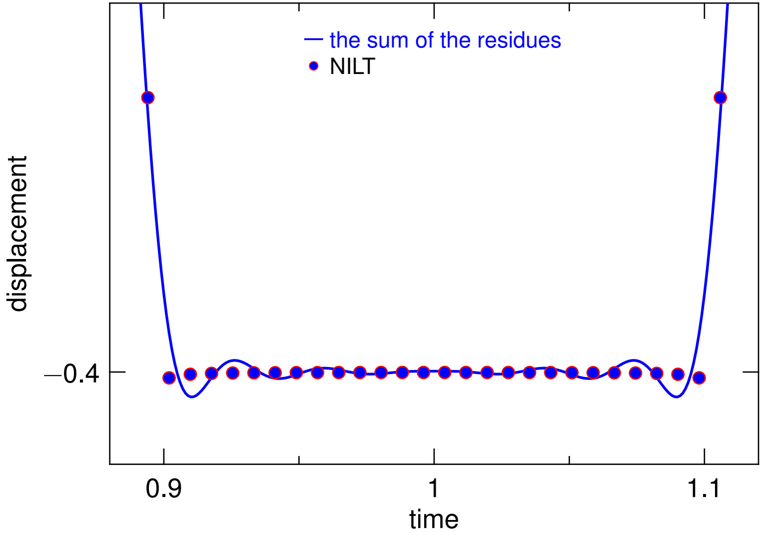
3 - The numerical inverse Laplace transform

- The FFT based NILT,
- ϵ-algorithm for accelerating convergence of the residual infinite series.

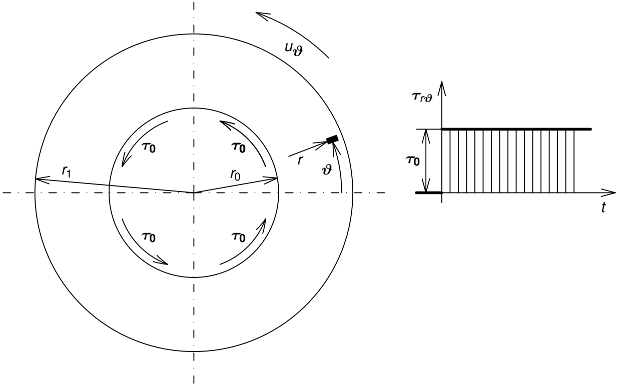
Brančík, L., Programs for fast numerical inversion of Laplace transforms in Matlab language environment, Proceedings of 7th Conference MATLAB'99, Prague, Czech Republic, 1999, pp. 27-39.



$x = 0.4, \qquad N = 50$



Problem of a torsionally loaded disc with concentric hole



Brepta, R., Okrouhlík, M., Valeš, F.,
Wave propagation and impact phenomena in solids and methods of solution, Academia, Prague, 1985 [in Czech].

- Wave equation

$$\frac{\partial^2 u_\vartheta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\vartheta}{\partial r} - \frac{u_\vartheta}{r^2} = \frac{1}{c_2^2}\frac{\partial^2 u_\vartheta}{\partial t^2}$$

- Initial conditions

$$u_\vartheta = 0, \quad \partial u_\vartheta/\partial t = 0$$

- Boundary conditions

$$r = r_0, \quad \tau_{r\vartheta} = -\tau_0$$

- The Laplace transform of wave equation

$$\frac{d^2\bar{u}_\vartheta}{dr} + \frac{1}{r}\frac{d\bar{u}_\vartheta}{dr} - \left(\frac{p^2}{c_2^2} + \frac{1}{r^2}\right)\bar{u}_\vartheta = 0$$

- $\bar{u}_\vartheta$:

$$\frac{\tau_0}{\left(\frac{ip^2}{c_2}\right)G} \cdot \frac{Y_2\left(\frac{ip}{c_2}r_1\right)J_1\left(\frac{ip}{c_2}r\right) - J_2\left(\frac{ip}{c_2}r_1\right)Y_1\left(\frac{ip}{c_2}r\right)}{Y_2\left(\frac{ip}{c_2}r_1\right)J_2\left(\frac{ip}{c_2}r_0\right) - J_2\left(\frac{ip}{c_2}r_1\right)Y_2\left(\frac{ip}{c_2}r_0\right)}$$

- $u_\vartheta / (\frac{r_1 \tau_0}{G})$:

$$\frac{2r_0^2/r_1^2}{(1-r_0^4/r_1^4)}\left(\frac{r}{r_1}\right)\left(\frac{c_2 t}{r_1}\right)^2 -$$

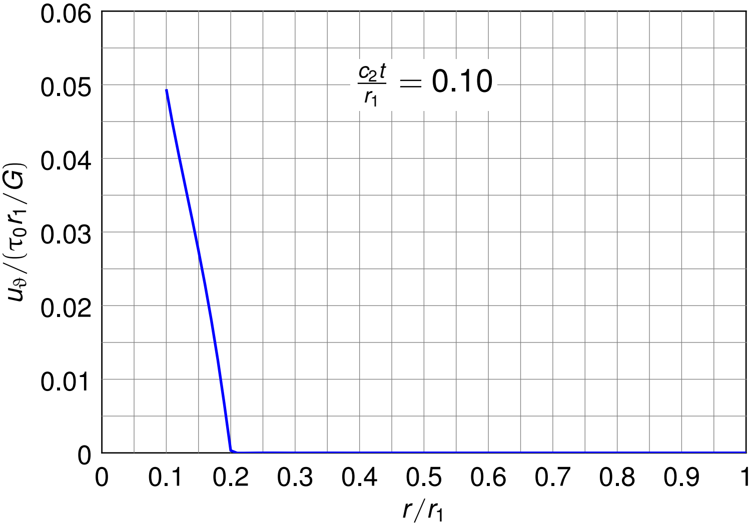
$$\left(\frac{r_0^2}{r_1^2}\right)\frac{1-r_0^2/r_1^2}{1+r_0^2/r_1^2}\left(\frac{r}{r_1}\right)\left[\frac{1}{3\,(1+r_0^2/r_1^2)} - \frac{1}{2(r/r_1)^2}\frac{\left(1-r^2/r_1^2\right)^2}{\left(1-r_0^2/r_1^2\right)^2}\right] -$$

$$\pi\sum_{n=1}^{\infty}\frac{Y_2(\xi_n r_0/r_1)}{\xi_n Y_2(\xi_n)}\frac{[Y_2(\xi_n)J_1(\xi_n r/r_1) - J_2(\xi_n)Y_1(\xi_n r/r_1)]}{\left[1-\frac{J_2^2(\xi_n r_0/r_1)}{J_2^2(\xi_n)}\right]}\cos\left(\xi_n\frac{c_2 t}{r_1}\right),$$

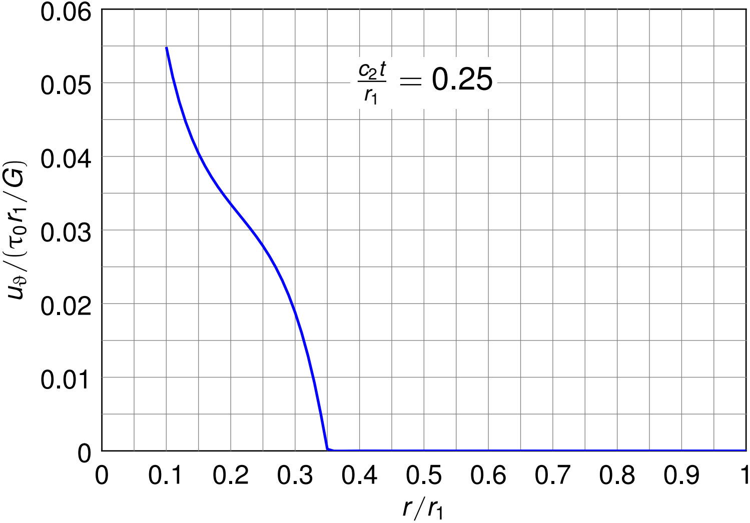
where ξ_n:

$$J_2\left(\xi_n\frac{r_0}{r_1}\right)Y_2(\xi_n) - J_2(\xi_n)Y_2\left(\xi_n\frac{r_0}{r_1}\right) = 0.$$

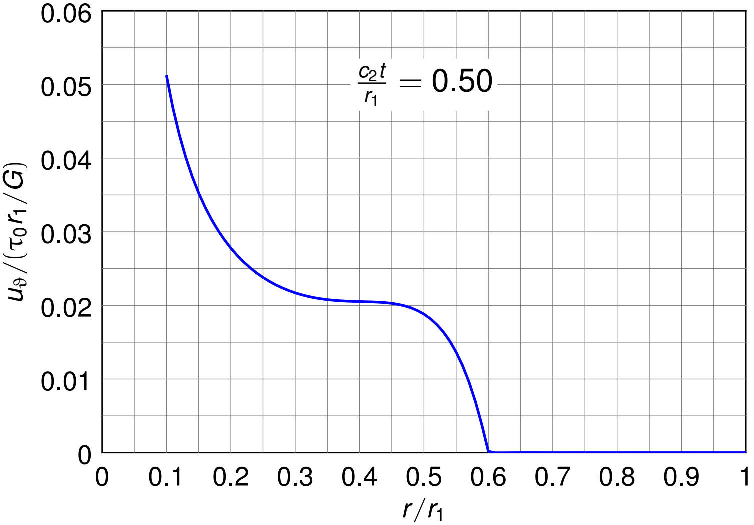
$r_0/r_1 = 0.1, \qquad N = 200$



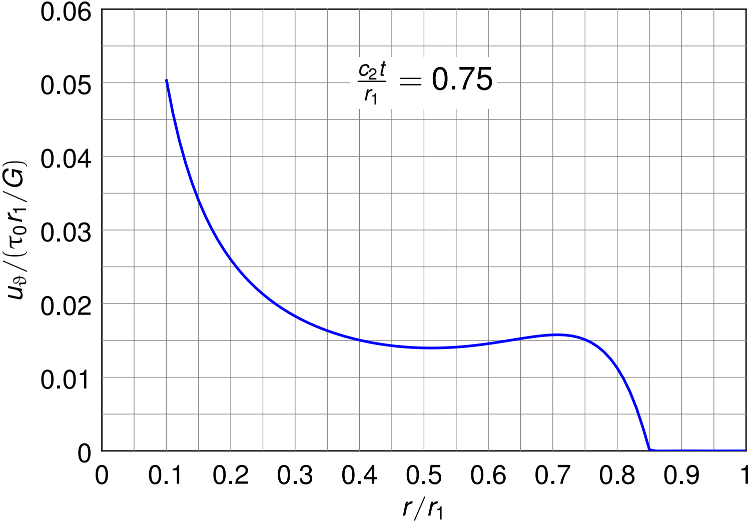
$$r_0/r_1 = 0.1, \qquad N = 200$$



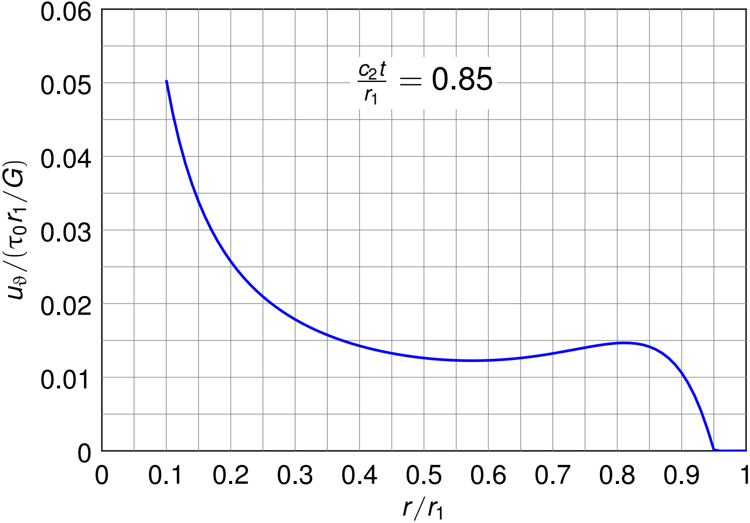
$r_0/r_1 = 0.1, \qquad N = 200$



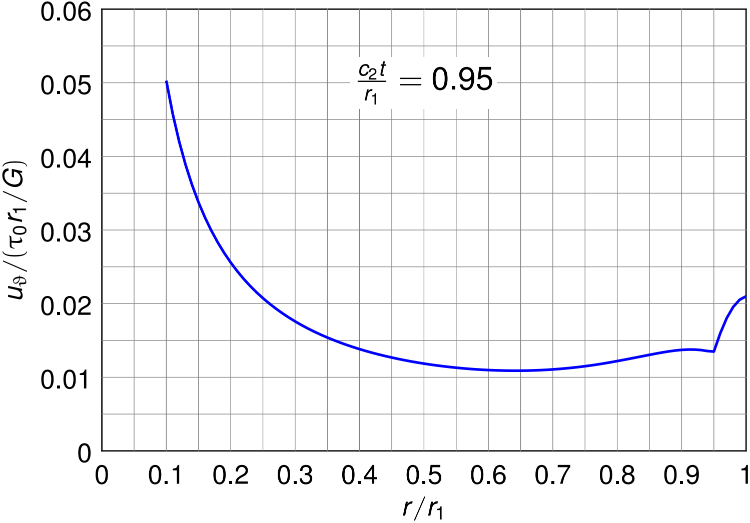
$$r_0/r_1 = 0.1, \qquad N = 200$$



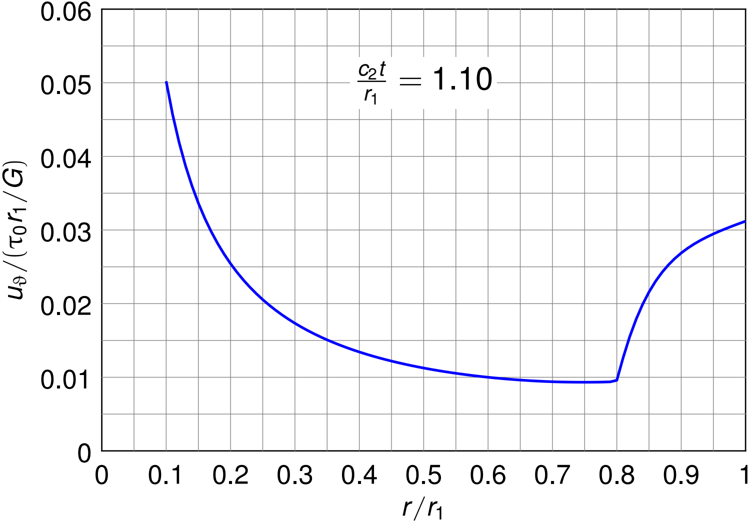
$$r_0/r_1 = 0.1, \qquad N = 200$$



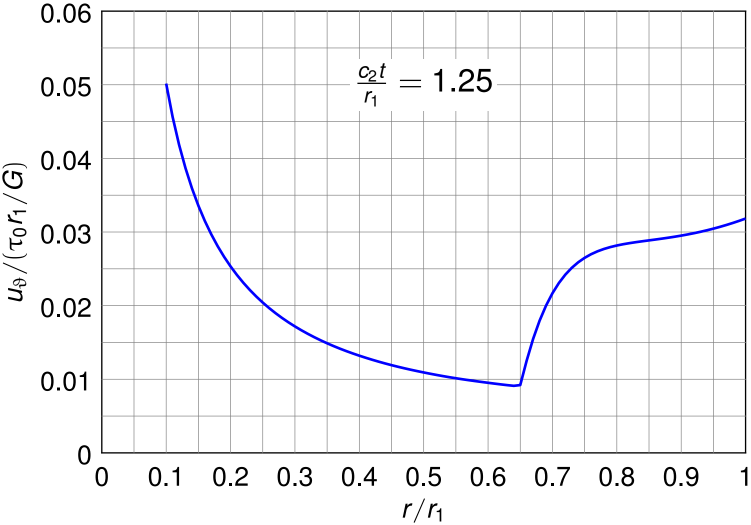
$r_0/r_1 = 0.1, \qquad N = 200$



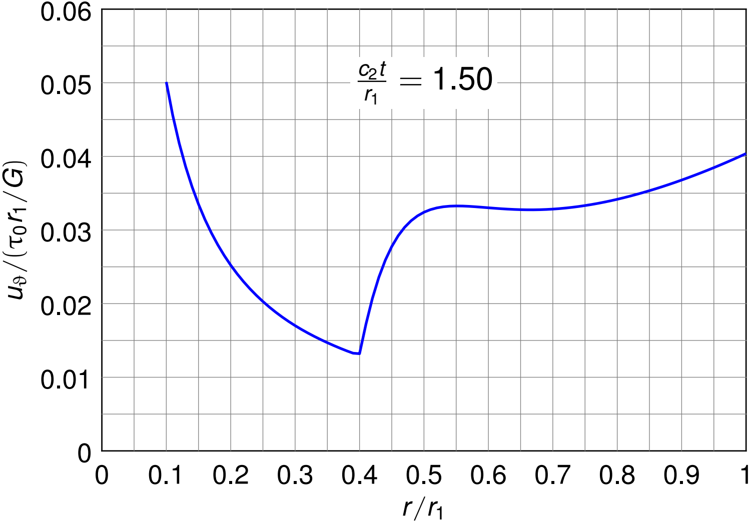
$$r_0/r_1 = 0.1, \qquad N = 200$$



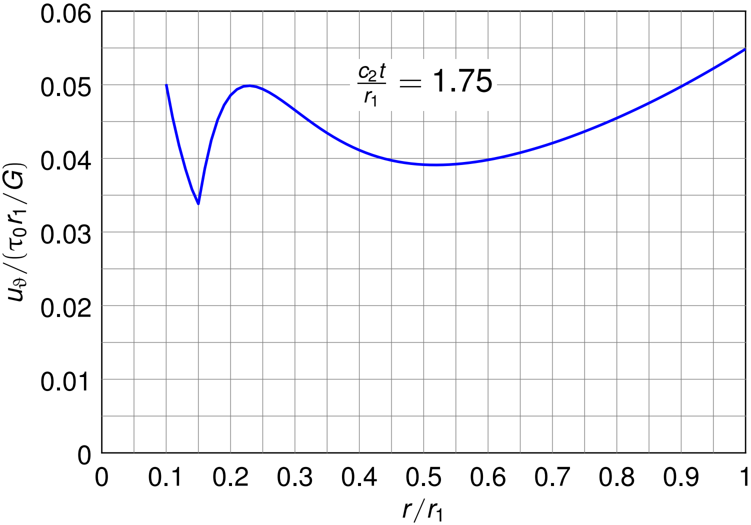
$r_0/r_1 = 0.1, \qquad N = 200$



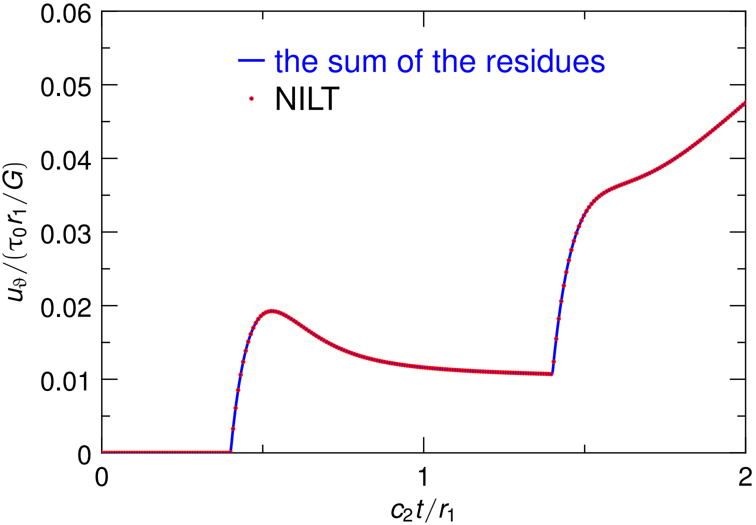
$r_0/r_1 = 0.1, \qquad N = 200$



$$r_0/r_1 = 0.1, \qquad N = 200$$



$r_0/r_1 = 0.1, \qquad r/r_1 = 0.5, \qquad N = 200$



- $\dot{u}_\vartheta/(\frac{c_2\tau_0}{G})$:

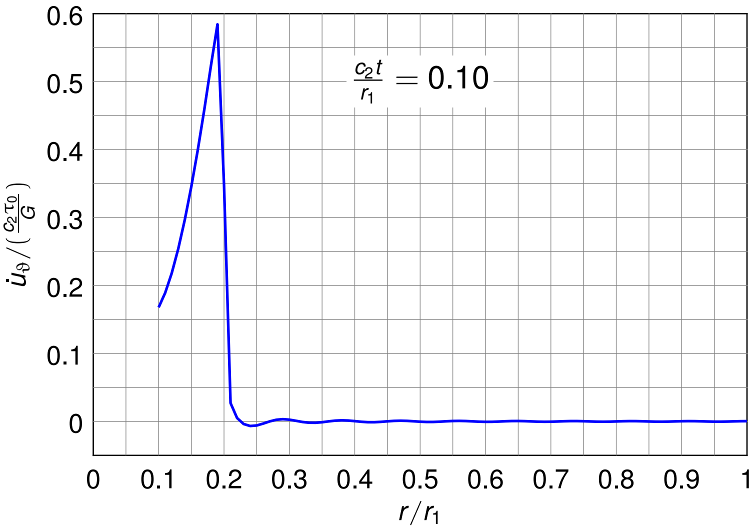
$$\frac{4r_0^2/r_1^2}{\left(1-\frac{r_0^4}{r_1^4}\right)}\left(\frac{r}{r_1}\right)\left(\frac{c_2t}{r_1}\right)+$$

$$\pi\sum_{n=1}^{\infty}\frac{Y_2\left(\xi_n\frac{r_0}{r_1}\right)}{Y_2(\xi_n)}\frac{\left[Y_2(\xi_n)J_1\left(\xi_n\frac{r}{r_1}\right)-J_2(\xi_n)Y_1\left(\xi_n\frac{r}{r_1}\right)\right]}{\left[1-\frac{J_2^2\left(\xi_n\frac{r_0}{r_1}\right)}{J_2^2(\xi_n)}\right]}\sin\left(\xi_n\frac{c_2t}{r_1}\right),$$

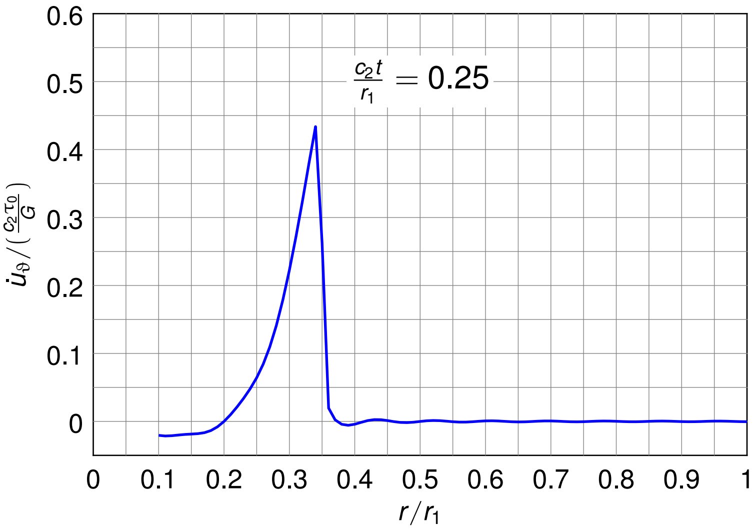
where ξ_n:

$$J_2\left(\xi_n\frac{r_0}{r_1}\right)Y_2(\xi_n)-J_2(\xi_n)Y_2\left(\xi_n\frac{r_0}{r_1}\right)=0.$$

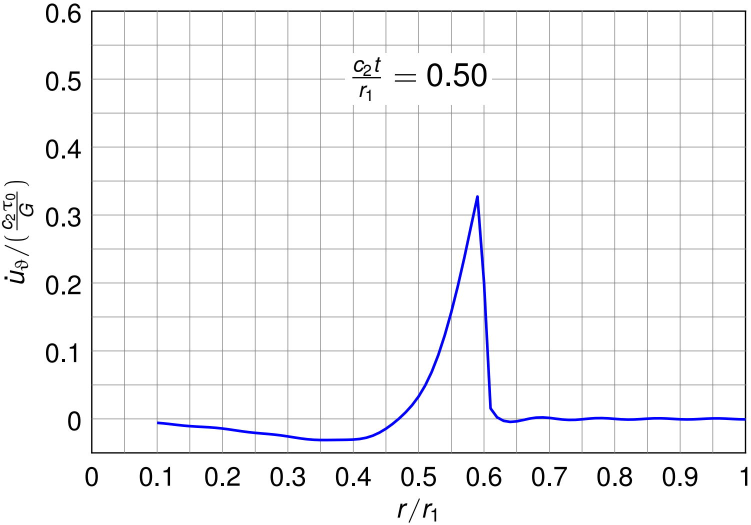
$$r_0/r_1 = 0.1, \qquad N = 200$$



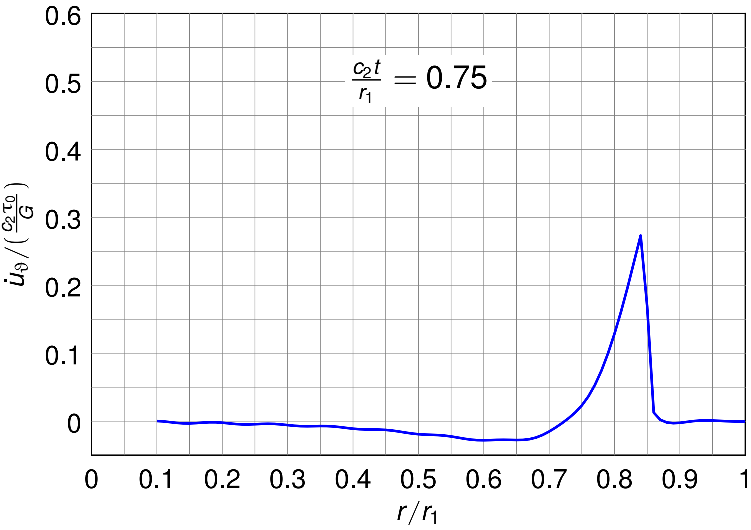
$\mathrm{r}_0/\mathrm{r}_1 = 0.1, \qquad \mathrm{N} = 200$



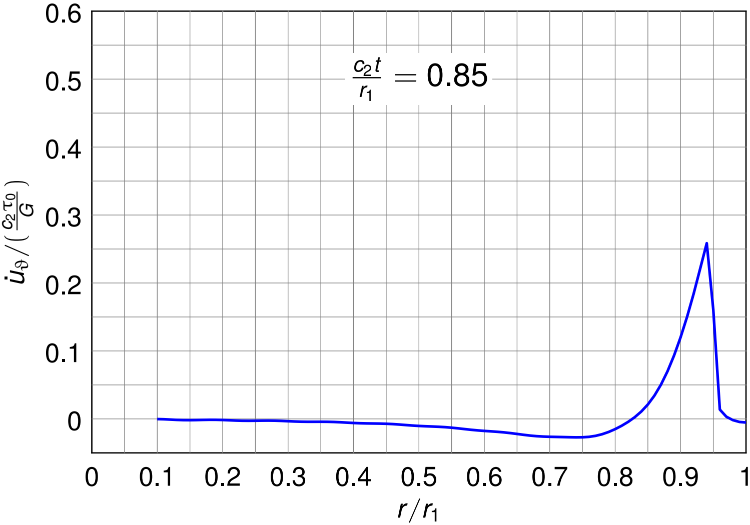
$$r_0/r_1 = 0.1, \qquad N = 200$$



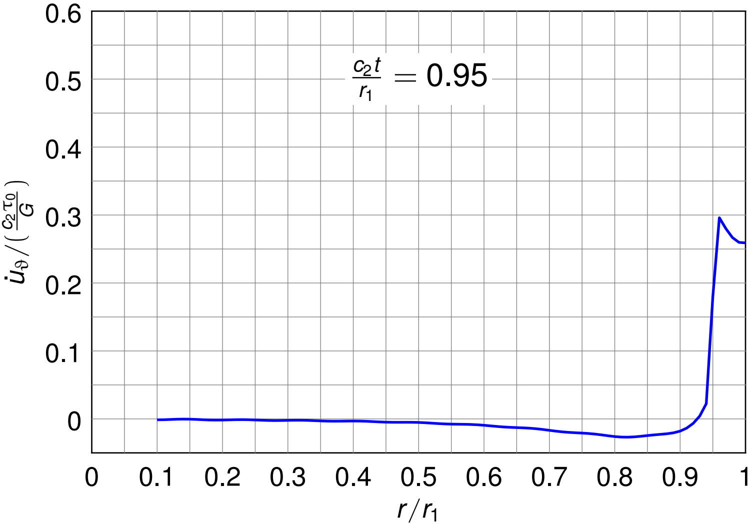
$$r_0/r_1 = 0.1, \qquad N = 200$$



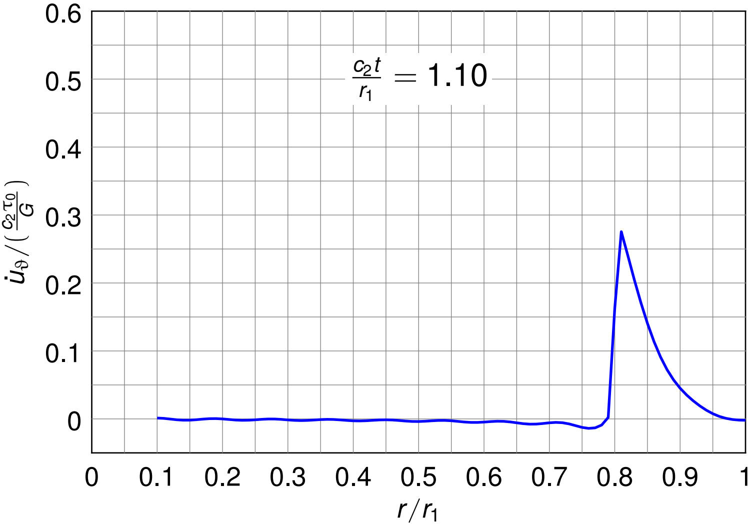
$r_0/r_1 = 0.1, \qquad N = 200$



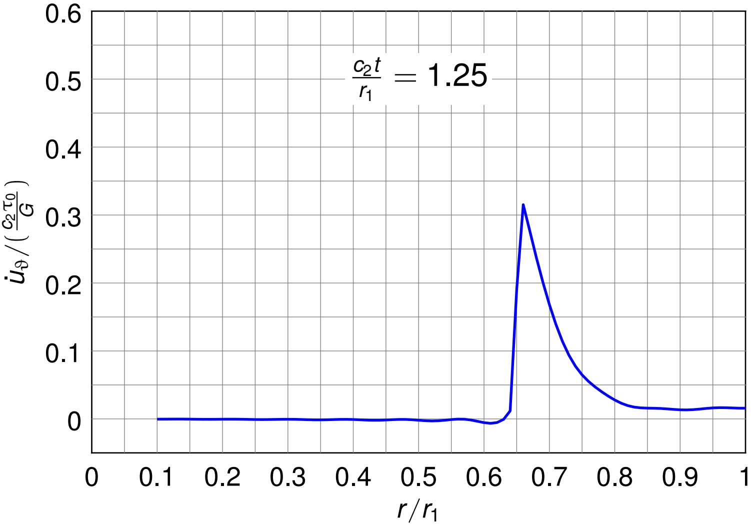
$$r_0/r_1 = 0.1, \qquad N = 200$$



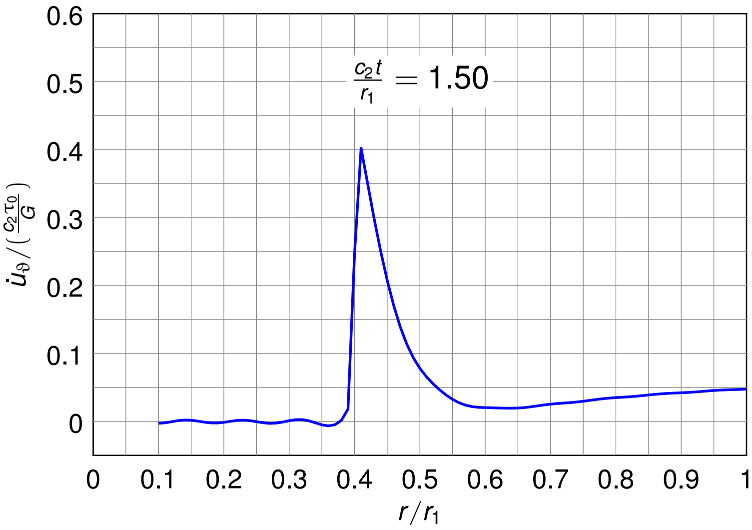
$r_0/r_1 = 0.1, \qquad N = 200$



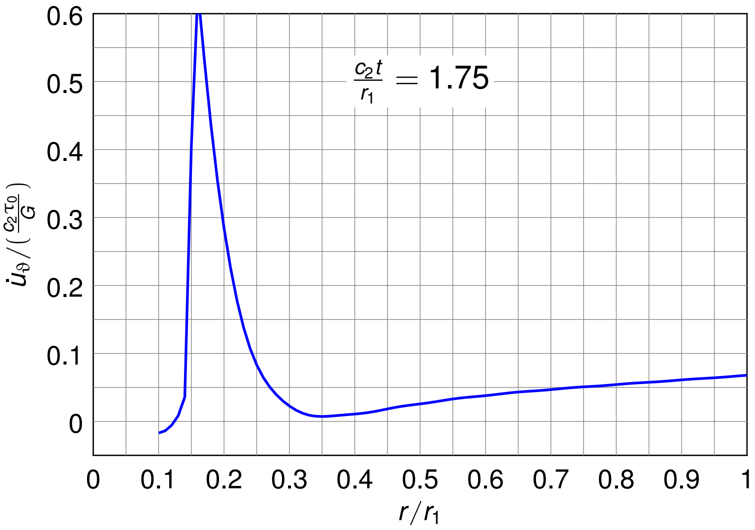
$$r_0/r_1 = 0.1, \qquad N = 200$$



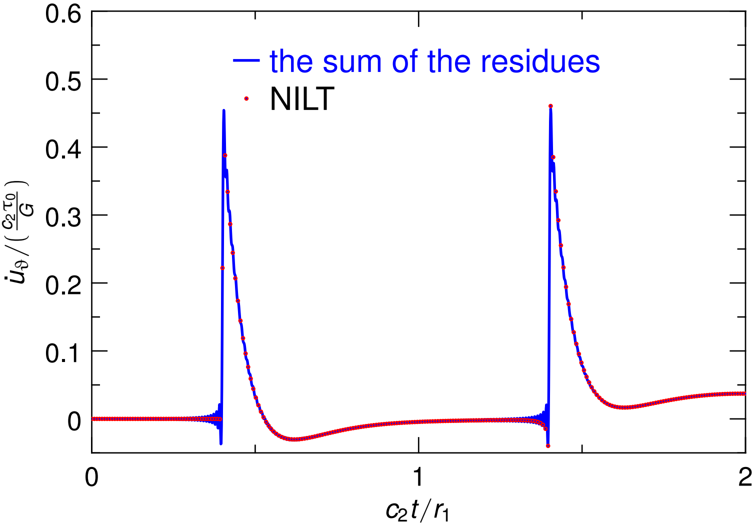
$$r_0/r_1 = 0.1, \qquad N = 200$$



$r_0/r_1 = 0.1, \qquad N = 200$



$r_0/r_1 = 0.1, \qquad r/r_1 = 0.5, \qquad N = 200$



- $\tau_{r\vartheta}/\tau_0$:

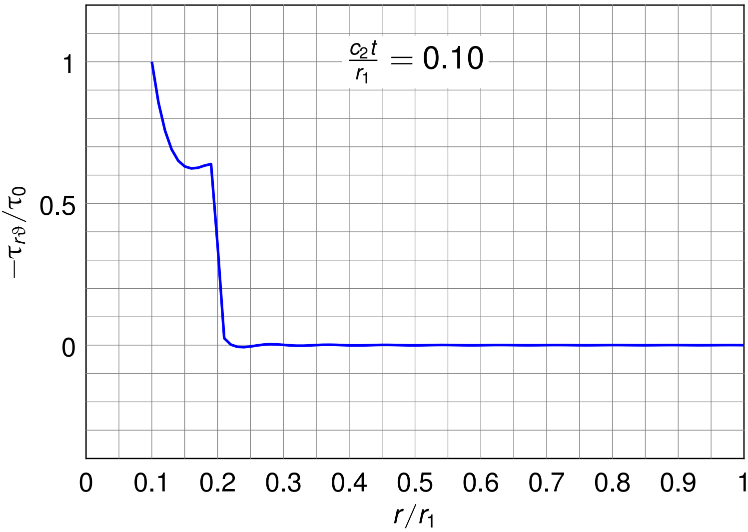
$$-\frac{\left(1-\frac{r^4}{r_1^4}\right)}{\left(1-\frac{r_0^4}{r_1^4}\right)}\frac{\left(\frac{r_0^2}{r_1^2}\right)^2}{\left(\frac{r}{r_1}\right)^2}+$$

$$\pi\sum_{n=1}^{\infty}\frac{Y_2\left(\xi_n\frac{r_0}{r_1}\right)}{Y_2(\xi_n)}\frac{\left[Y_2(\xi_n)J_2\left(\xi_n\frac{r}{r_1}\right)-J_2(\xi_n)Y_2\left(\xi_n\frac{r}{r_1}\right)\right]}{\left[1-\frac{J_2^2\left(\xi_n\frac{r_0}{r_1}\right)}{J_2^2(\xi_n)}\right]}\cos\left(\xi_n\frac{c_2t}{r_1}\right),$$

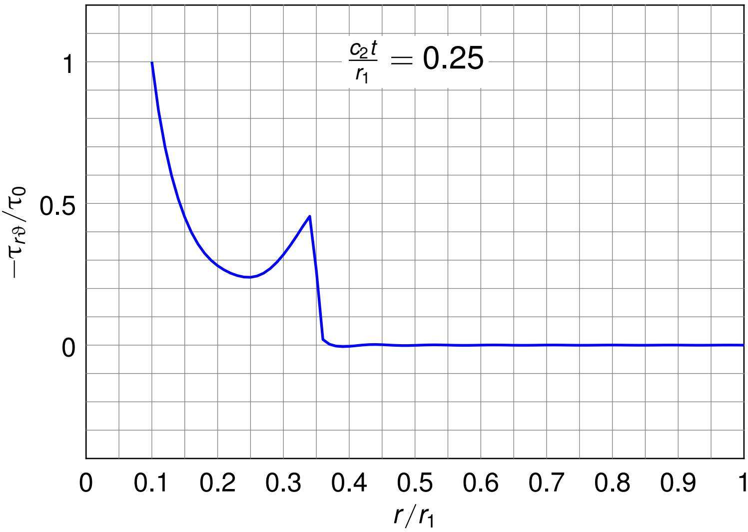
where ξ_n:

$$J_2\left(\xi_n\frac{r_0}{r_1}\right)Y_2(\xi_n)-J_2(\xi_n)Y_2\left(\xi_n\frac{r_0}{r_1}\right)=0.$$

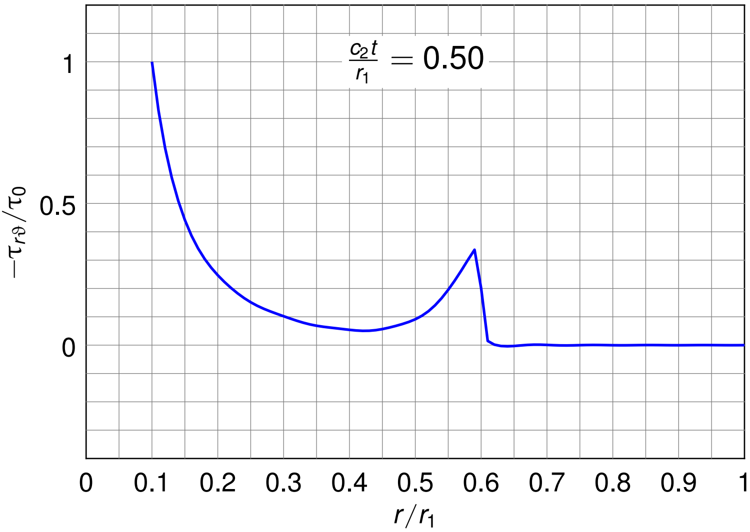
$$r_0/r_1 = 0.1, \qquad N = 200$$



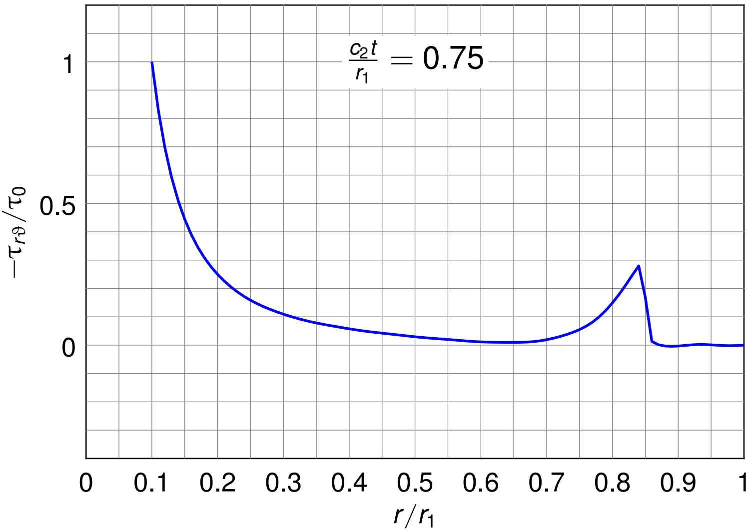
$r_0/r_1 = 0.1, \qquad N = 200$



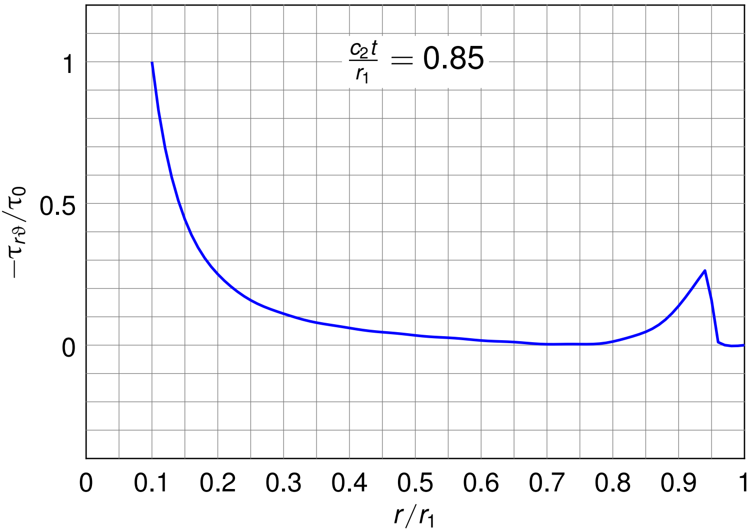
$r_0/r_1 = 0.1, \qquad N = 200$



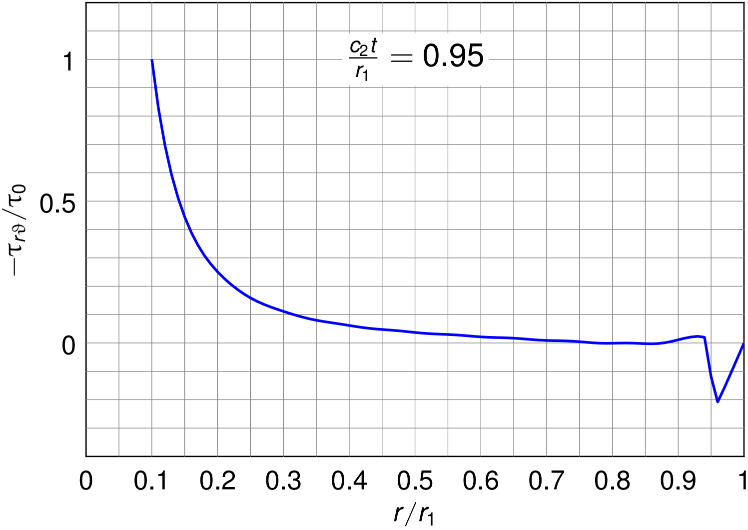
$r_0/r_1 = 0.1, \qquad N = 200$



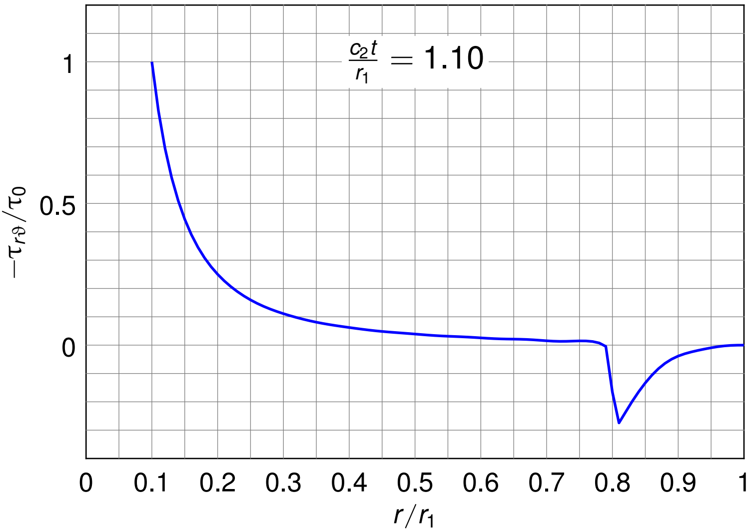
$$r_0/r_1 = 0.1, \qquad N = 200$$



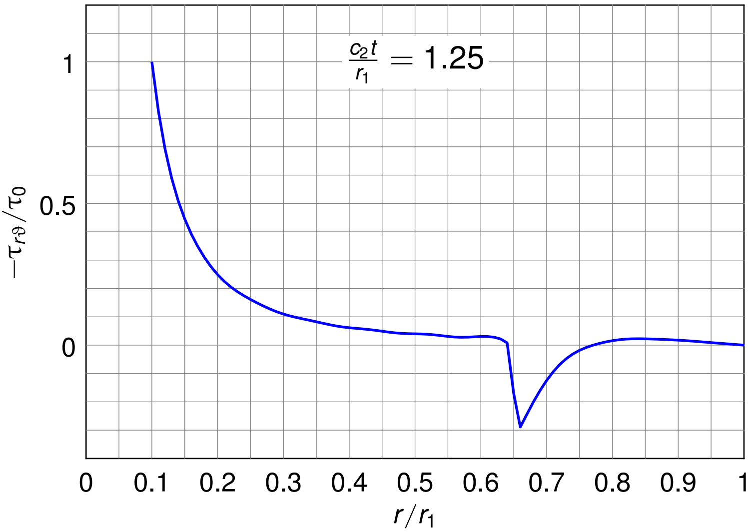
$\mathrm{r}_0/\mathrm{r}_1 = 0.1, \qquad \mathrm{N} = 200$



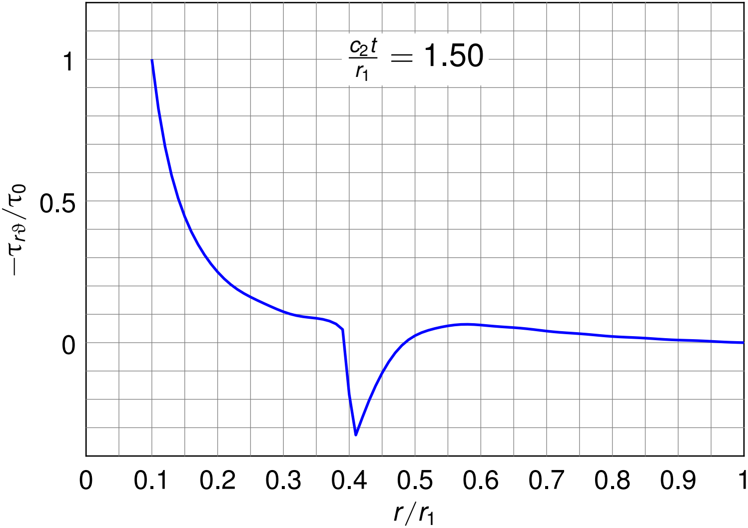
$$r_0/r_1 = 0.1, \qquad N = 200$$



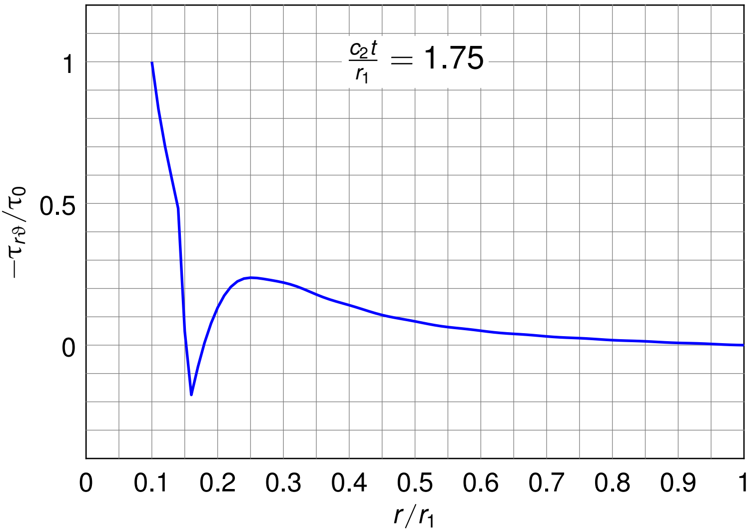
$r_0/r_1 = 0.1, \qquad N = 200$



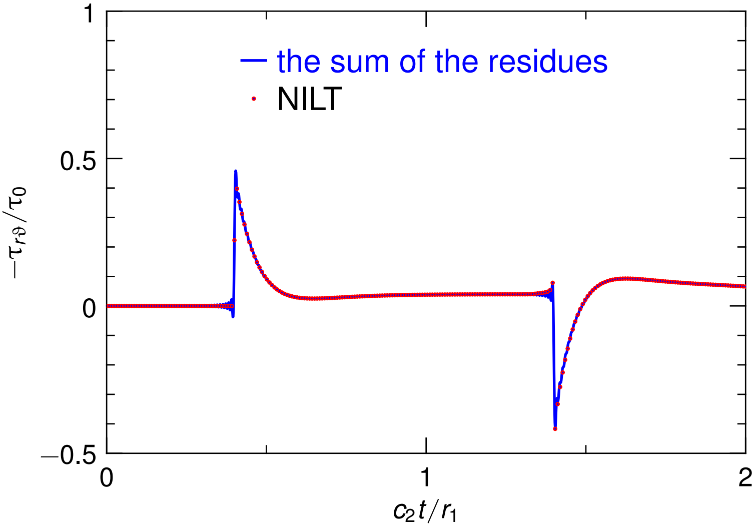
$$r_0/r_1 = 0.1, \qquad N = 200$$



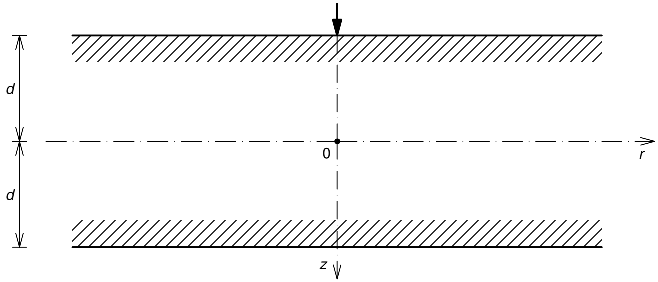
$r_0/r_1 = 0.1, \qquad N = 200$



$r_0/r_1 = 0.1, \qquad r/r_1 = 0.5, \qquad N = 200$



Problem of a point loaded thick plate



Valeš, F., Report Z847/83, IT CAS, Prague, 1983 [in Czech].

Valeš, F., Report Z887/84, IT CAS, Prague, 1984 [in Czech].

- Wave equation (cylindrical coordinates)
- Initial conditions
- Boundary conditions
- The Laplace transform ⇒ reduction of t
- The Hankel transform ⇒ reduction of r
- The inverse Laplace transform:
    - a sum residues
    - Fubini's theorem

$$\bar{u}_z = \frac{1}{2G}\int_0^\infty \left(-\frac{F_4}{pL} + \frac{G_4}{pT}\right) k_1 a(\gamma) J_0(\gamma r)\, d\gamma,$$

where

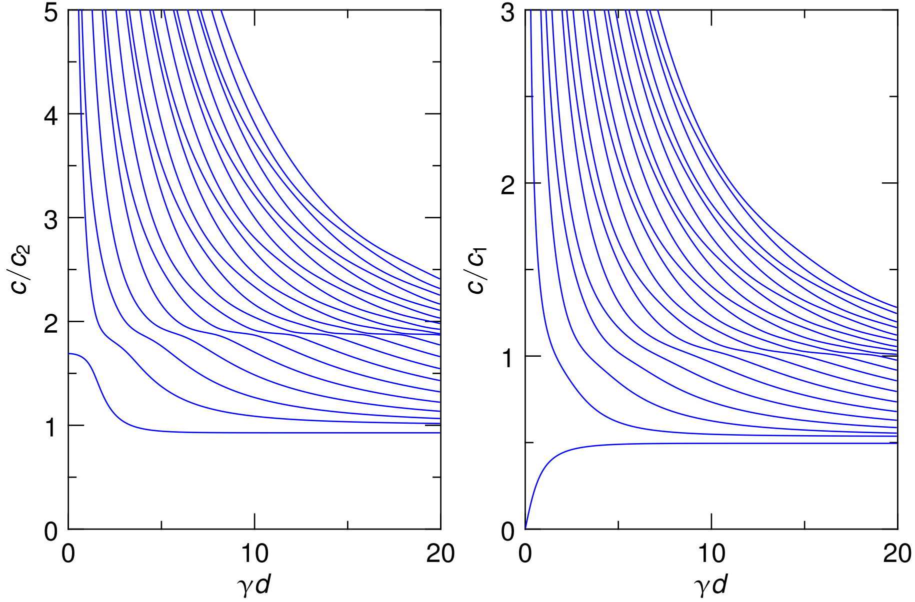
$$F_4 = \left[2 + \left(\frac{p}{c_2\gamma}\right)^2\right] \sinh(k_1.\gamma z)\sinh(k_2.\gamma d) - 2\sinh(k_1.\gamma d)\sinh(k_2.\gamma z),$$

$$G_4 = \left[2 + \left(\frac{p}{c_2\gamma}\right)^2\right] \cosh(k_1.\gamma z)\cosh(k_2.\gamma d) - 2\cosh(k_1.\gamma d)\cosh(k_2.\gamma z),$$

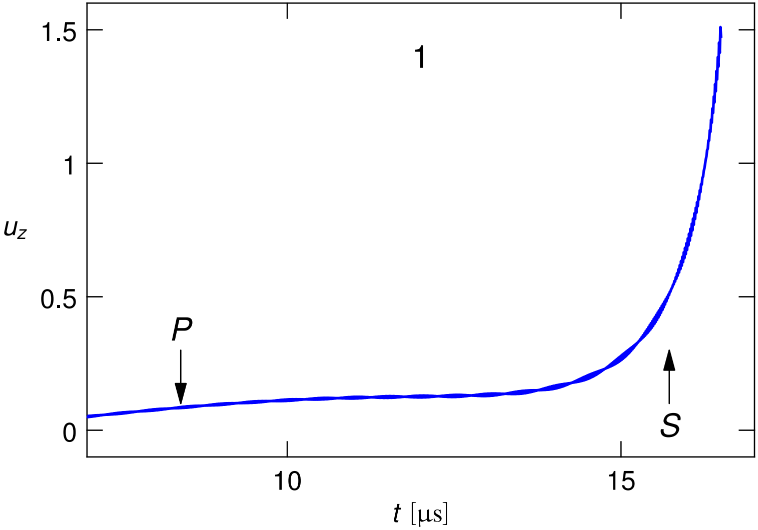
$$L = \left[2 + \left(\frac{p}{c_2\gamma}\right)^2\right]^2 \cosh(k_1.\gamma d)\sinh(k_2.\gamma d) - 4k_1k_2\sinh(k_1.\gamma d)\cosh(k_2.\gamma d),$$

$$T = \left[2 + \left(\frac{p}{c_2\gamma}\right)^2\right]^2 \sinh(k_1.\gamma d)\cosh(k_2.\gamma d) - 4k_1k_2\cosh(k_1.\gamma d)\sinh(k_2.\gamma d),$$

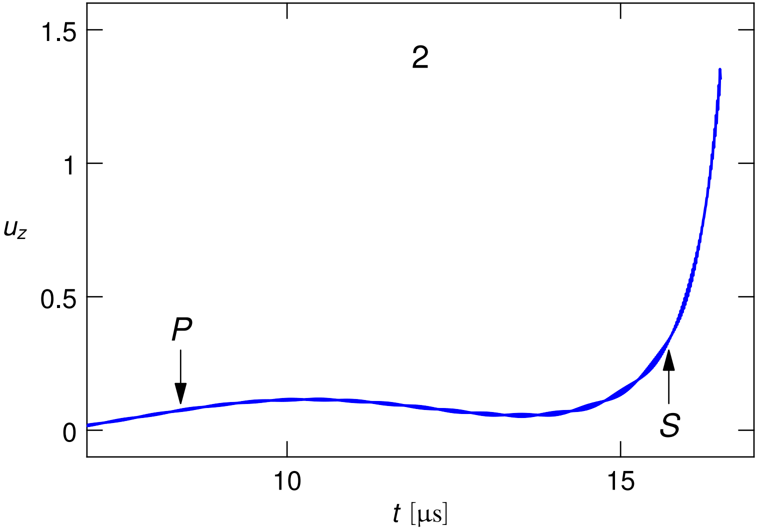
$$k_1 = \sqrt{1 + \left(\frac{p}{c_1\gamma}\right)^2}, \quad k_2 = \sqrt{1 + \left(\frac{p}{c_2\gamma}\right)^2}.$$


5
4
3
2
1
0
0
10
20
c/c_2
γd
3
2
1
0
0
10
20
c/c_1
γd

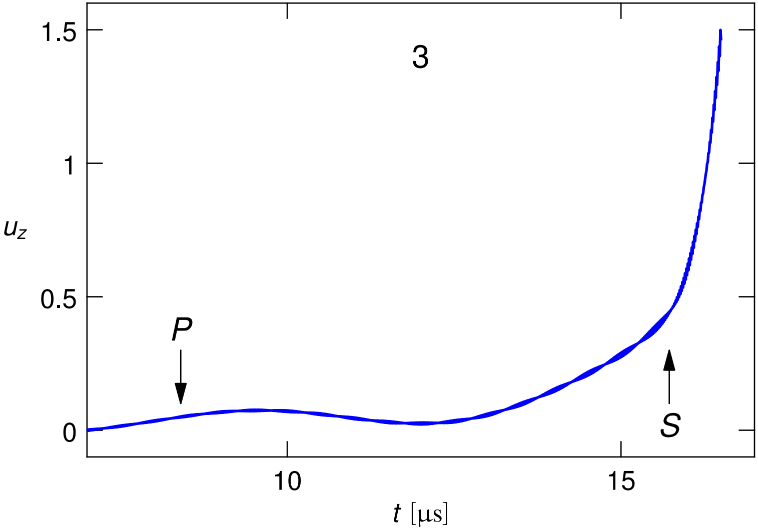
$d = 25$ mm, $r = 50$ mm, surface



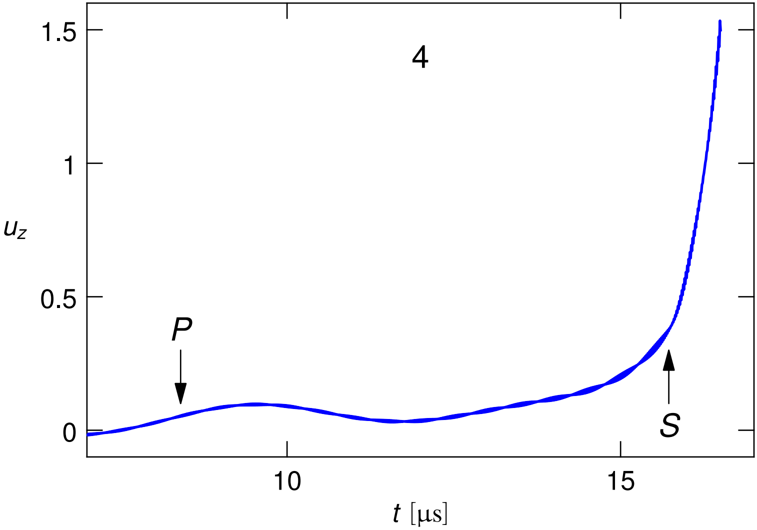
$d = 25$ mm, $r = 50$ mm, surface



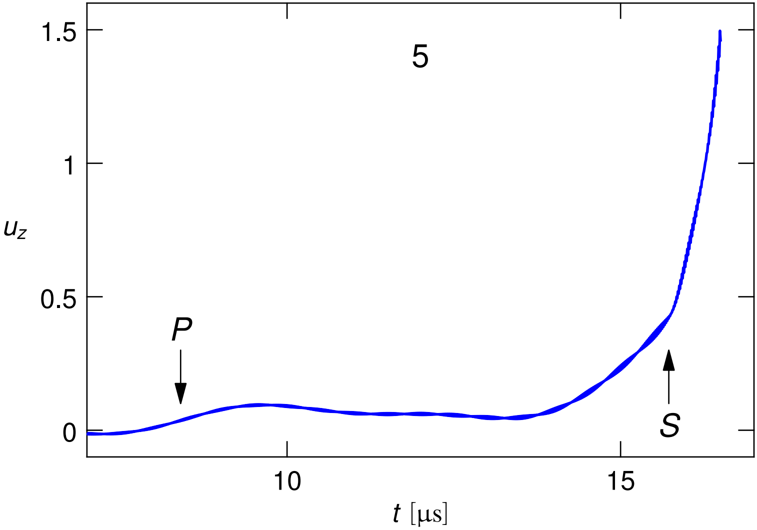
$d = 25$ mm, $r = 50$ mm, surface



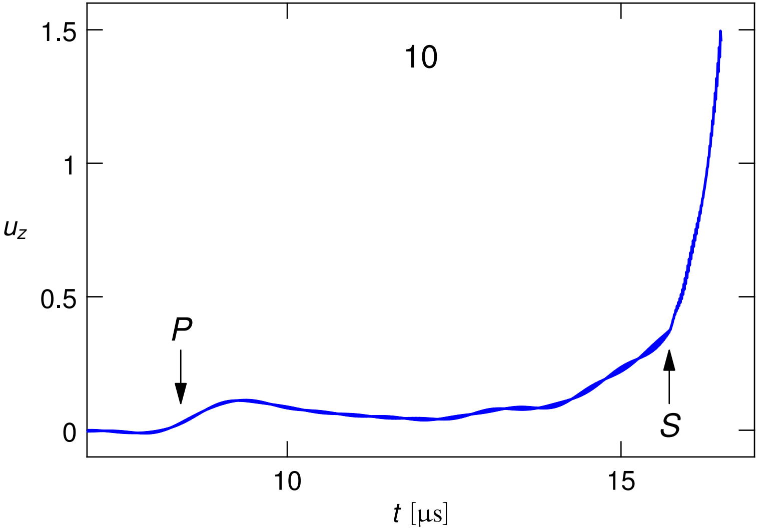
$\mathrm{d} = 25$ mm, $\mathrm{r} = 50$ mm, surface



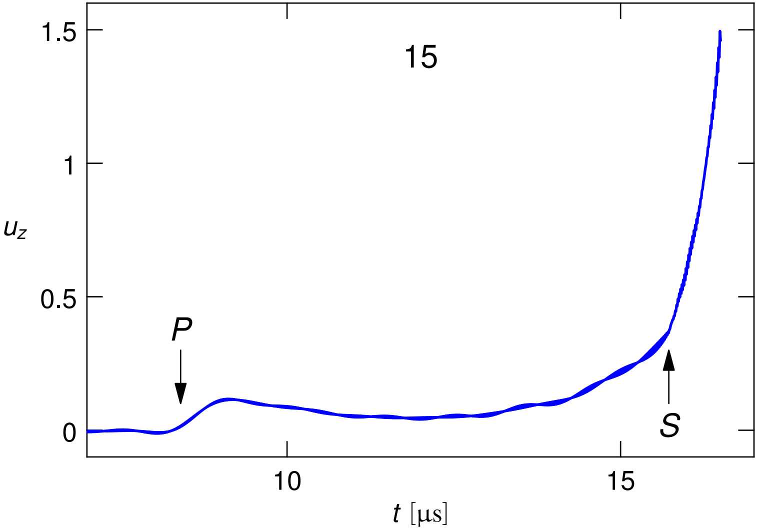
$d = 25$ mm, $r = 50$ mm, surface



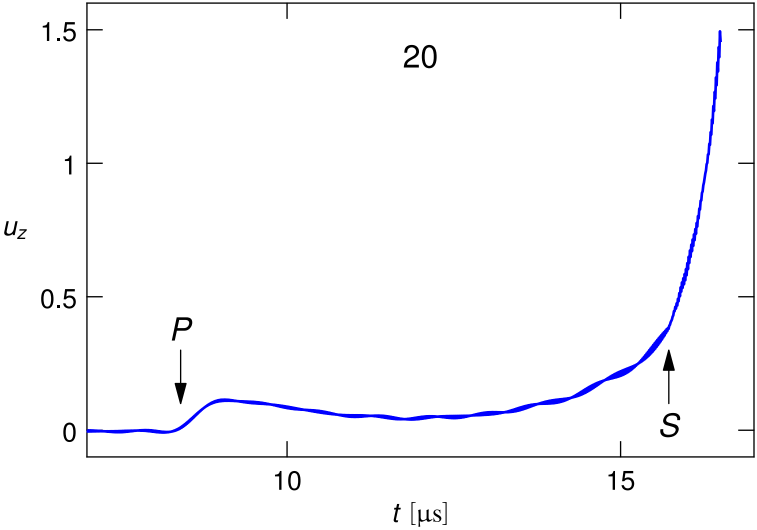
$\mathrm{d} = 25$ mm, $\mathrm{r} = 50$ mm, surface



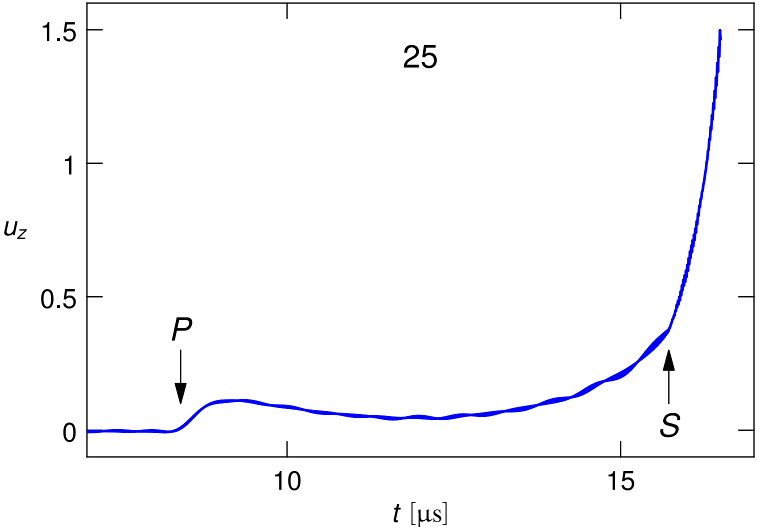
$d = 25$ mm, $r = 50$ mm, surface



$\mathrm{d} = 25$ mm, $\mathrm{r} = 50$ mm, surface



$\mathrm{d} = 25$ mm, $\mathrm{r} = 50$ mm, surface



Generalized ray theory

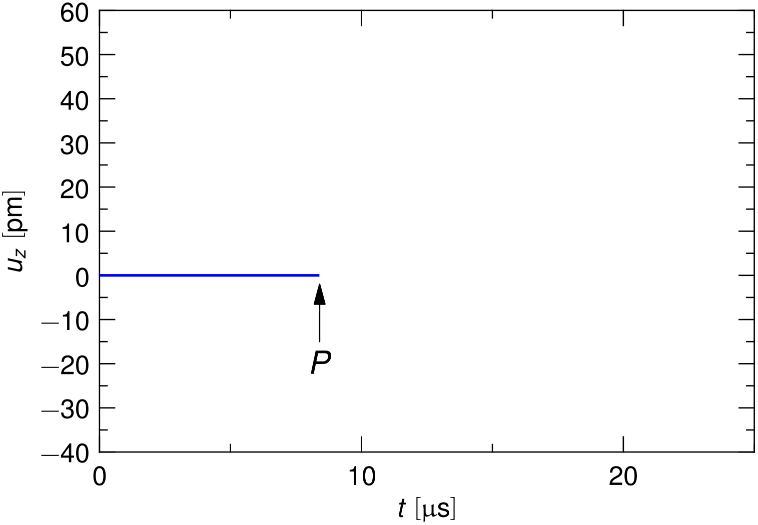
- This method is based on the *Bromwich expansion method*.
- The ray integrals for transient waves (given from *Bromwich expansion*) are evaluated by applying the so-called *Cagniard's method*.
- Spencer - the concept of *generalized ray path*. He showed how the ray integrals can be constructed directly from the known source functions and reflection and transmission coefficients for plane waves along each path.
- The solution is exact up to the time of arrival of the next ray.

Pao, Y.-H., Gajewski, R.R., The generalized ray theory and transient responses of layered elastic solids, "Physical Acoustics" ed. Warren P. Mason and R.N. Thurston, Academic Press, New York, Vol. 13, 1977.

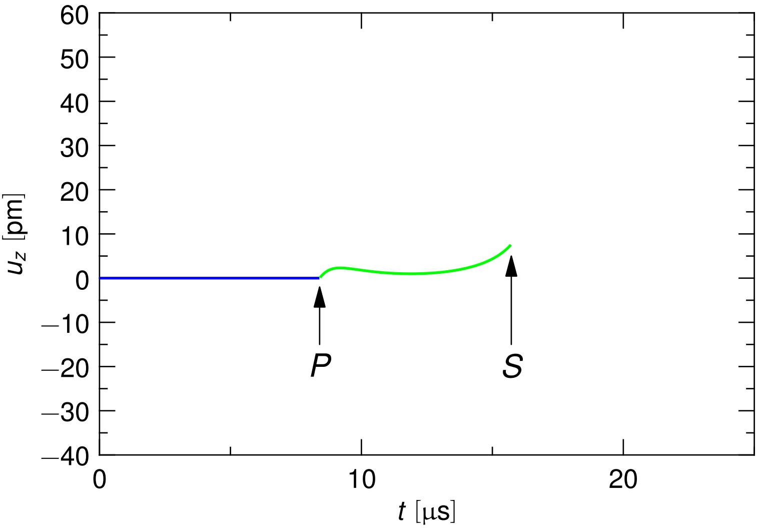
Cagniard, L., Reflection and Refraction of Progressive Seismic Waves, McGraw-Hill, New York, 1962.

half-space, $r = 50$ mm, surface

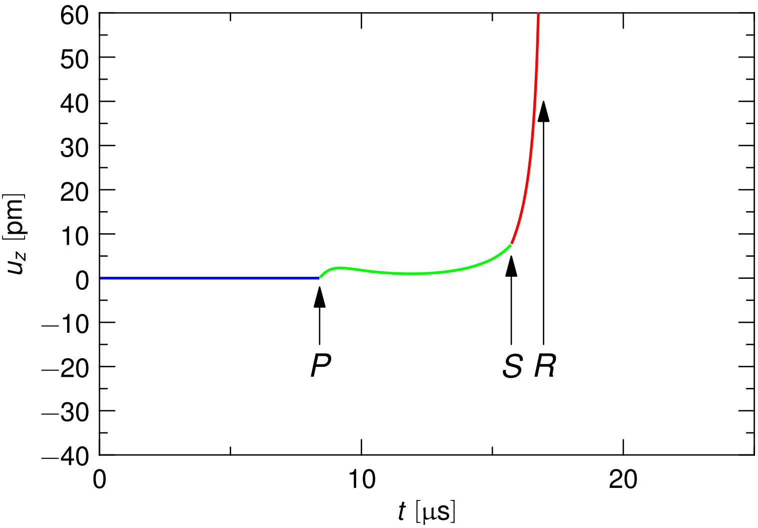
half-space, $r = 50$ mm, surface



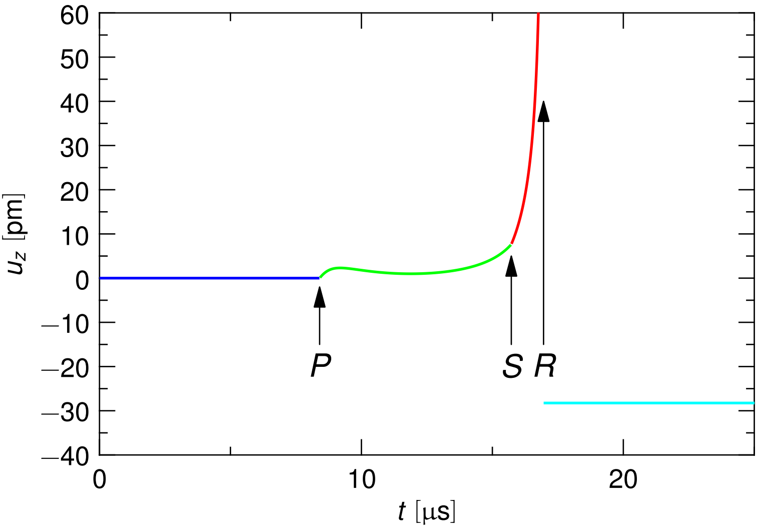
half-space, $r = 50$ mm, surface



half-space, $r = 50$ mm, surface



half-space, $\mathrm{r} = 50$ mm, surface



Conclusion

Sum residues

The numerical work involved in this analysis is long and difficult.
The method is more effective for long-time transient responses at remote observation points.

Generalized ray theory

The solution is exact up to the time of arrival of the next ray.
The method is more effective for short-time transient responses at near observation points.

Numerical inverse Laplace transform

The method is very efficient, but error is hard estimated.

Thank you for your attention!

Any questions?

The work was supported by the grant GA CR No. 101/09/1630 and by the institutional support RVO: 61388998.

Contents